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UV RADIATION INDUCED SYNTHESIS OF AMINO ACIDS FROM A MIXTURE OF GASES IN PRESENCE OF WATER

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ABSTRACTS

Methane, ammonia, carbon dioxide, nitrogen and hydrogen were bubbled through water contained in a transparent quartz vessel kept under an ordinary mercury quartz lamp. After 5 hours co-chromatographic analysis of the solution through which these gases were continuously passed indicated formation of amino acids. Experiment was continued for 5 hours a day for 10 days and amongst the amino acids formed glycine, alanine, lysine, arginine, amino butyric acid, tyrosine, tryptophane, aspartic and glutamic acid could be identified. Gases used in these experiments were those generally believed to be present in the prebiological reducing atmosphere of the Earth, and therefore, the results obtained throw light on the abiogenic formation of amino acids in the pre-aetualistic atmosphere era of the Earth when there was no ozone layer to cut off the ultraviolet component of sunlight.

INTRODUCTION

Work on individual phototysis of methane, ammonis, carbon dioxide, nitrogen and hydrogen has been reviewed in a general way by Ritchie[1], Ellis and Wells[2] Noyes and Leighton[3], McDowell[4], Steacie[5] and Borrel,[6] and it is believed that such photolysis must have played the primary step in the conversion of primitive substances of the Earth to biogenic meterial.[7-14]

It has been observed that carbon dioxide dissolved in water on exposure to light soon gets converted to formaldehyde,[15-20] Reid[21] has investigated the action of ultraviolet light from a mercury quartz burner on aqueous solutions of formalde-

hyde and ammonia through which carbon dioxide was bubled. He found large quantities of hexamethylene tetramine formed at pH > 7·5. He could also detect the formation of amino acids like glycine alanine and possibly also histidine. At pH 6, the could identify glycine, alanine and possibly also glutamic acid and leucine. According to him the amino acids are formed by the photodecomposition of hexamethylene tetramine. Further, he noticed that if the bubbling of carbon dioxide was stopped much lesser amounts of amino acids could be detected.

From the foregoing it would appear that there is a distinct possibility of formation of aminoacids in detectable amounts if methane, ammonia, nitrogen, carbon dioxide and hyrogen are bubbled through water contained in a transparent quarts vessel kept under an ordinary quartz mercury lamp. Both carbon dioxide and methane can act as sources of carbon as well as formaldehyde. Formaldehyde could react with ammonia to give hexamethylene tetramine which is known to yield amino acids under these conditions. Hydrogen could furnish the reducing atmosphere. Once traces of amino acids are obtained they could be expected to multiply as indicated by our earlier work.[12,22,23,24,25]

The present paper records attempts to achieve the synthesis of amino acids under the above conditions.

EXPERIMENTAL

All glass and quartz vessels used in these experiments were sterilised by heating them in an electric oven at 250°C for 2 hours. Water employed was double glass distilled. It was sterilised in an autoclave at 20 lbs./sq. inch steam pressure for 20 minutes before use.

Gases used were prepared as follows:

Methane ; was prepared by heating a mixture of anhydrous pure fused sodium acetate and sodalime in a strong pyrex vessel. The gas was bubbled through a trap containing water before use.

Carbon dioxide : was obtained by adding dilute hydrochloric acid over calcium carbonate. It was bubbled through water before use.

Nitrogen : was obtained by warming slowly a mixture of pure ammonium chloride and sodium nitrite. Thh gas was bubbled through traps containing dilute sulphuric acid and water before use.

Ammonia : was obtained be slowly heating liquor ammonia.

Hydrogen : was prepared by the action of pure dilute sulphuric acid on well washed zinc. It was bubbled through a trap containing water before use.

About 300 ml of water was taken in a transparent quartz long necked flask. The gases were led in the water by glass tubes. The rate of bubbling of gases was kept very low-just sufficient to keep the solution agitated. The whole vessel was kept exposed to radiation from a 250 watt low pressure quartz mercury vapour lamp kept at a distance of about 30 cms. The lamp had a reflector and was placed above the reaction vessel. The experiment was performed for about 5 hours a day. It was continued in this manner on subsequent days. During the course of the experiments the temperature of the water generally remained in the range 30-35°C. After each day the mouth of the flask was covered by a thin polythene sheet which was opened the next day before commencing the experiment.

At intervals a small portion of the solution was taken out and analysed chromatographically by methods used by us earlier.[23,24,25] Either the solution was used as such employing several drops at a point on the chromatographic paper kept under a small hot air blower, or, the solution was concentrated under high vacuum and then used. In either case similar results were obtained. Preliminary

analysis was carried out by using both one dimensional or two dimensional paper chromatography taking 18½″ × 22½″ sheet of Whatman No. 1 filter paper. The spots obtained after spraying with ninhydrin solution were tentatively identified. Final confirmation was done by co-chromatography using pure samples of tentatively identified amino acids. Results are recorded in table II.

After 50 hours exposure the apparatus was dismantled and the remaining solution was concentrated under high vacuum. A small amount of white residue was obtained. This was dissolved in water and analysed chromatographically.

It may also be mentioned that as the time of exposure increased the concentrations of the animo acids also increased as evidenced by intensity of spots on the chromatograms.

Every time before the chromatographic analysis, a part of the solution was tested for all types of microbes employing the media whose compositions are recorded in Table 1. The details concerning clearing, adjustment of reaction, tubing sterilisation and cultivation of various microbes in different media were similar to those described earlier.[26,27,28,29]. Every time the solutions were found to be perfectly sterile.

TABLE I

Analysis of irradicted samples for microbial contaminations.

Name of culture media	Nutrient-Dextro-agar	Glucose Pepton e-agar	Potato-Dextro-seagar	Czapek's sucrose-Nitrate solution
Composition	Distilled water—1000 ml. Agar—17 g. Beef extract—3 g. Peptone—10 g. Dextrose—10 g.	Distilled water—1000 ml. Agar—17 g. Peptone—10 g. Dextrose—40 g.	Distilled water—1000 ml. Agar—17 g. Sliced Potato—200 g. Dextrose—20 g.	Distilled water—1000 ml. Sodium nitrate—2 g. Potassium dihydrogen phosphate—1·00.g Potasium chloride—0·50 g. Magnesium sulphate—0·50 g. Ferrous sulphate—0·01 g. Sucrose— 30 g.
Results	No growth or chromogenesis was observed in this medium with solutions tested	No growth or chromogenesis was observed in this medium with any of the solutions tested	No growth or chromogenesis was observed in this medium with any of the solutions tested	No growth or chromogenes is was observed in this culture with any of the solutions tested.

TABLE II

Exposure time in hours	Result of chromatographic analysis for amino acids
0	No ninhydrin positive spot
5	Glycine, alanine, n-amino-butyric acid and a very faint spot corresponding to tyrosine
10	Glycine, alanine, n-amino-butyric acid, tyrosine, lysine and a few other very faint spots
15	Glycine, alanine, n-amino butyric acid, tyrosine, lysine, arginine+two faint spots correspondirg with aspartric and glutamic acids.
20	ditto
25	ditto
30	ditto
40	ditto +two faint spots corresponding with tryptophane and histidine
45	ditto
50	Glycine, alanine, n-amino butyric acid, tyrosine, lysine, arginine, aspartic acid, glutamic acid, tryptophane and a faint spot corresponding to histidine (tentative).

DISCUSSION

When methane, ammonia, carbon dioxide, nitrogen and hydrogen are bubbled through water contained in a transparent quartz vessel kept under a mercury quartz lamp for 5 hours glycine, alanine and n-amino-butyric acid could be identified in the mixture. After 10 hours of exposure lysine and tyrosine could also be trade in the mixture. Further increase of exposure time by 5 hours showed the formation of arginine besides other amino acids mentioned earlier. After exposure of 20 hours formation of aspartic and glutamic acid was also indicated. Further exposure to a period of 10 more hours showed increase in the amount of these acids as evidenced by increasing intensity of spots on the chromatograms. After a total exposure of 40 hours there was indication of formation of tryptophane and histidine(?). Exposure was continued further for 10 hours but formation of no other new amino acid was indicated.

These preliminary experiments in which no microbial contamination could be detected indicate that it is possible that amino acids could have been formed easily from methane, ammonia carbon dioxide, nitrogen and hydrogen which are known to be present in the primitive anoxygenic atmosphere of the Earth. In the absence of oxygen the sunlight reaching the Earth would be very rich in the ordinary ultraviolet as there would be no ozone layer to cut it off. These experiments clearly point out that under heterogenous conditions the formation of amino acids in the anoxygenic prebiological atmosphere of the Earth from gases present would have presented not much difficulty.

Further work to confirm these results is in progress.

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CHEMICAL EXAMINATION OF THE PLANT "FAGONIA CRETICA" LINN : STUDY OF THE FAT

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ABSTRACT

The composition of the fatty acid of the fat obtained from the plant 'Fagonia cretica '[1,2] has been determined. The fatty acids after fractionation into solid and liquid fractions have been further fractionated into their respective components by the urea adduct formation method. The components found are Palmitic 30.37 %, stearic 50.74 % and oleic 17.46 %. The unsaponifiable matter has been found to contain *n*-triacontanol and β sitosterol.[3]

EXPERIMENTAL

About 5 kg. of dried and powdered plant were extracted with alcohol. The alcoholic extract after concentration to half of its volume was kept over night when a deposit was obtained. The deposit was filtered and studied separately. The filtrate was concentrated and the residue obtained was extracted with ether and then with petroleum ether. The petroleum ether extract was purified with animal charcoal, and finally passed through a column of fuller's earth. The solvent was then distilled off. A semi- solid fat (135 gm.) of light gray colour was obtained. This gave the following constants:

Acid value 8·7 : Saponification value 197·5:

Sodine value 18·8 : Unsaponifiable matter 1·5 %

A known amount of fat (127 gm.) was saponified and the fatty acids (86·2 gm.) were recovered by the usual procedure. The mixed fatty acids (82·5 gm. saponification value 201·8, Iodine value 22·3) were segregated into solid and liquid fractions by Twitchel Lead Salt Alcohol process[4] modified by Hilditch and co-workers[5]. They were found to have the following constants :

Fraction	Weight	Saponification value	Iodine value
Solid	56·7 gm. (69·36%)	208·7	1·18
Liquid	25 05 gm. (30·64%)	199·7	43·7

The solid and the liquid acids were further fractionated by the urea adduct formation method using 6 grams of urea and 150 ml of saturated solution of urea in methanol: ethyl acetated (7:3) mixture at each stage.

(a) *Fractionation of solid acids with urea* :

Ten conical flasks (numbered S_1 to S_{10}), each of 250 ml capacity and having a ground glass stopper were taken. The flasks S_1 to S_9 were employed for the adduct formation, while the flask S_{10} for collecting raffinates. The fatty acids were recovered from each flask by treating the adduct in the respective flask with warm acidulated water followed by extraction with ether. The different fractions were weighed separately and their saponification and iodine values were determined. The result are recorded in Table I :

TABLE I

Fractionation of Solid acids with urea

Weight = 35.5 gm., Sap. value = 208.7, Iodine value = 1 18

Adduct	Wt. of the fraction	S.V.	M. wt.	I.V.	Palmitic	Stearic	Oleic
S_1	7.85 gm	201.4	278	0.85	1.67 gm	6.11 gm	0.07 gm
S_2	6.84 ,,	202.3	276.8	0.87	1.75 ,,	5.03 ,,	0.08 ,,
S_3	6.40 ,,	200.4	279.6	1.01	1.00 ,,	5.33 ,,	0.07 ,,
S_4	5.54 ,,	207.3	269	1.3	0.98 ,,	4.48 ,,	0.08 ,,
S_5	2.43 ,,	218.3	255.5	0.98	2.43 ,,	——	0.02 ,,
S_6	1.94 ,,	217.1	257.9	2.1	1.78 ,,	0.12 ,,	0.04 ,,
S_7	1.20 ,,	222.8	251.4	1.1	1.18 ,,	——	0.02 ,,
S_8	0.80 ,,	219.4	245.3	0.91	.79 ,,	——	——
S_9	0.20 ,,	219.1	259.3	——	——	——	——
Raffinate :							
S_{10}	1.71 ,,	209	268	25.38	0.71 ,,	0.52 ,,	0.42 ,,
Total wt. 34.93 gm					12.29 gm	21.59 gm	0.84 gm
% acid in solid fraction					35.42	62.16	2.42

A qualitative study of the solid acids was done by paper chromatography. Whatman No. 1 filter paper impregnated with 10% solution of liquid paraffin in benzene and acetic acid : water (7 : 1) as solvent were used. The chromatograms after development were heated at 80-110°C. and were immersed in 1000 ml of water containing 20 ml of a saturated solution of copper acetate. The paper was washed with water containing 0.01% acetic acid, and was then dipped into a

solution of 1.5% aqueous potassium ferrocyanide[6,7]. The acids present in the solid fraction were found to be palmitic and stearic acids.

(b) *Fractionation of liquid acids with urea* :

Eighteen conical flasks (numbered L_1 to L_{18}) were taken. The flasks L_1—L_9 were employed for adduct formation and the flasks L_{13}-L_{18} for collecting raffinate fractions. The results are recorded in Table 2 :

TABLE 2

Fractionation of liquid acid with urea

Weight = 23.28 gm ; sap. value = 199.7, Iodine valve = 43.7

Adducts	weight	S.V.	M. wt.	I.V.	Palmitic	stearic	Oleic
L_1	3.60 gm	197.8	283.1	38.4	0.08 gm	2.09 gm	1.43 gm
L_2	3.15 ,,	203.6	275.1	37.7	0.59 ,,	1.25 ,,	1.31 ,,
L_3	2.97 ,,	209.4	276.5	51.3	0.76 ,,	0.52 ,,	1.69 ,,
L_4	1.8 ,,	212.2	263.9	46.5	0.63 ,,	0.24 ,,	0.93 ,,
L_5	1.57 ,,	209.2	268	49.8	0.40 ,,	0.30 ,,	0.87 ,,
L_6	1.23 ,,	208.5	268.5	49.3	0.34 ,,	0.26 ,,	0.63 ,,
L_7	1.12 ,,	211.1	265.3	53.4	0.32 ,,	0.15 ,,	0.65 ,,
L_8	0.30 ,,	——	——	59.8	——	——	0.20 ,,
L_9	0.16 ,,	——	——	61.7	——	——	0.11 ,,
Raffinate							
L_{18}	2.1 ,,	207.2	270.1	49.2	0.48 ,,	4.48 ,,	1.14 ,,
L_{17}	1.83 ,,	209.3	267.6	48.6	0.49 ,,	0.36 ,,	0.98 ,,
L_{10}	1.23 ,,	204.6	273.6	65.5	0.14 ,,	0.21 ,,	0.88 ,,
L_{11}	0.88 ,,	215.9	259.5	63.8	0.23 ,,	0.3 ,,	0.62 ,,
L_{12}	0.37 ,,	——	——	67.1	——	——	0.20 ,,
L_{13}	0.28 ,,	——	——	——	——	——	——
Total wt. 22.59 gm.					4.36 gm	5.89 gm	11.64gm
% acid in liquid fraction					20	26.8	53.18

A qualitative study of the liquid acid fraction was done by paper chromatography using Whatman No. 1 filter paper impregnated with 2 % olive oil (Kobrle 1954)[8], and the chromatogram was developed by using 75% ethanol as solvent. The developed chromatogram was then treated with a saturated solution of copper acetate followed by 1.5% aqueous solution of potassium ferrocyanide.

With the help of Table 1 and 2 the percentage of each acid in mixed fatty acid has been found to be palmitic 30.37 %, stearic 50.47 %, and oleic 17.46 %.

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INTEGRALS INVOLVING THE H-FUNCTION

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ABSTRACT

The aim of this paper is to evaluate two definite integrals involving Fox's H-function [3; p. 408]. Since the H-function is a generalization of Meijer's G-function [1; p. 207], some of the results obtained resently for the G function are extended by our results.

1. *Introduction*: Following the definition of FOX with a slight change in the parameters we define

$$H_{\gamma,\delta}^{\alpha,\beta}\left[x\left|\begin{matrix}(a_1,l_1),\ldots,(a_\gamma,l_\gamma)\\(b_1,f_1),\ldots(b_\delta,f_\delta)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{\alpha}\Gamma(b_j-f_j\xi)\prod_{j=1}^{\beta}\Gamma(1-a_j+l_j\xi)}{\prod_{j=\alpha+1}^{\delta}\Gamma(1-b_j+f_j\xi)\prod_{j=\beta+1}^{\gamma}\Gamma(a_j-l_j\xi)}x^{\xi}\,d\xi \tag{1}$$

where an empty product is interpreted as 1, $O \leqslant \alpha \leqslant \delta$, $O \leqslant \beta \leqslant \gamma$; $l_1,\ldots,l_\gamma$ and $f_1,\ldots,f_\delta$ are all positive. Also the parameters are such that no pole of $\Gamma(b_j-f_j\xi)$, $j=1,2,\ldots,\alpha$ coincides with any pole of $\Gamma(1-a_j+l_j\xi)$, $j=1,2,\ldots,\beta$ and the integral on the right hand side of (1) is convergent. These assumptions will be retained throughout this note.

The H-function is an analytic function of x; it is symmetric in the ordered pairs $(a_1,l_1),\ldots,(a_\beta,l_\beta)$ likewise in $(a_{\beta+1},l_{\beta+1}),\ldots,(a_\gamma,l_\gamma)$ in $(b_1,f_1),\ldots,(b_\alpha,f_\alpha)$ and in $(b_{\alpha+1},f_{\alpha+1}),\ldots,(b_\delta,f_\delta)$.

Now we give two formulae for the H-function [4, p. 4]

(*i*)

$$H_{\gamma,\delta}^{\alpha,\beta}\left[x^{-1}\left|\begin{matrix}(a_1,l_1),\ldots,(a_\gamma,l_\gamma)\\(b_1,f_1),\ldots,(b_\delta,f_\delta)\end{matrix}\right.\right]=H_{\delta,\gamma}^{\beta,\alpha}\left[x\left|\begin{matrix}(1-b_1,f_1),\ldots,(1-b_\delta,f_\delta)\\(1-a_1,l_1),\ldots,(1-a_\gamma,l_\gamma)\end{matrix}\right.\right] \tag{2}$$

(*ii*)

$$H_{\gamma+2,\delta+1}^{\alpha,\beta+2}\left[x\left|\begin{matrix}(\alpha_1,r_1),(\alpha_2,r_2),(a_1,s),\ldots,(a_\gamma,s)\\(b_1,s),\ldots,(b_\delta,s),(\beta_1,r_3)\end{matrix}\right.\right]$$

$$=(2\pi)^{(1-s)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)+\frac{1}{2}(r_3-r_1-r_2+1)}\,s^{\sum_{j=1}^{\delta}(b_j)-\sum_{j=1}^{\gamma}(a_j)+\frac{1}{2}(\gamma-\delta)}\,r_1^{\frac{1}{2}-\alpha_1}\,r_2^{\frac{1}{2}-\alpha_2}$$

$$\times\, r_3^{\beta_1-\frac{1}{2}}\,G_{\gamma s+r_1+r_2,\ \delta s+r_3}^{\alpha s,\ \beta s+r_1+r_2}\left[\frac{x\,s^{s(\gamma-\delta)}\,r_1^{r_1}\,r_2^{r_2}}{r_3^{r_3}}\left|\begin{matrix}\Delta(r_1,\alpha_1),\Delta(r_2,\alpha_2),\Delta(s,a_1),\ldots,\Delta(s,a_\gamma)\\\Delta(s,b_1),\ldots,\Delta(s,b_\delta),\Delta(r_3,\beta_1)\end{matrix}\right.\right] \tag{3}$$

where $\triangle(l,a)$ stands for the quantities $\frac{a}{l}, \frac{a+1}{l}, \ldots, \frac{a+l-1}{l}$, l being a positive integer.

To obtain (3) we express the H-function into its equivalent contour integral with the help of (1) and then apply the formula namely

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)} m^{mz-\frac{1}{2}} \prod_{j=0}^{m-1} \Gamma\left(z+\frac{j}{m}\right) \tag{4}$$

(3) reduces to the following result when $s=r_1=r_2=r_3=n$ by virtue of [6, p. 401].

$$H^{\alpha,\beta}_{\gamma,\delta}\left[x \,\middle|\, \begin{matrix}(a_1,n),\ldots,(a_\gamma,n)\\(b_1,n),\ldots,(b_\delta,n)\end{matrix}\right] = \frac{1}{n} G^{\alpha,\beta}_{\gamma,\delta}\left[x^{1/n} \,\middle|\, \begin{matrix}a_1,\ldots,a_\gamma\\ b_1\ldots,b_\delta\end{matrix}\right] \tag{5}$$

where n is a positive integer.

2. The following results will be required during the proofs [2, pp. 398, 399].

$$\int_0^1 x^{l-1}(1-x)^{\beta-\gamma-n}\,{}_2F_1(-n,\beta;\gamma;x)\,dx = \frac{\Gamma(\gamma)\,\Gamma(l)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma+n-l)}{\Gamma(\gamma+n)\,\Gamma(\gamma-l)\,\Gamma(\beta-\gamma+l+1)} \tag{6}$$

where $n = 0, 1, 2, \ldots$; $R(\beta-\gamma) > n-1$ and $R(l) > 0$.

$$\int_0^1 x^{l-1}(1-x)^{\beta-l-1}\,{}_2F_1(\alpha,\beta;\gamma;x)\,dx = \frac{\Gamma(\gamma)\,\Gamma(l)\,\Gamma(\beta-l)\,\Gamma(\gamma-\alpha-l)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-l)} \tag{7}$$

where $R(l) > 0$, $R(\beta-l) > 0$ and $R(\gamma-\alpha-l) > 0$.

3. First integral :

$$\int_0^1 x^{l-1}(1-x)^{\beta-\gamma-n}\,{}_2F_1(-n,\beta;\gamma;x)\,H^{p,q}_{r,s}\left[zx^m \,\middle|\, \begin{matrix}(a_1,l_1),\ldots,(a_r,l_r)\\(b_1,f_1),\ldots,(b_s,f_s)\end{matrix}\right]dx$$

$$= \frac{\Gamma(\gamma)\Gamma(\beta-\gamma+1)}{\Gamma(\gamma+n)} H^{p+1,q+1}_{r+2,s+2}\left[z \,\middle|\, \begin{matrix}(1-l,m),(a_1,l_1),\ldots,(a_r,l_r),(\gamma-l,m)\\(\gamma+n-l,m),(b_1,f_1),\ldots(b_s,f_s),(\gamma-\beta-l,m)\end{matrix}\right]$$

where $n=0, 1, 2,\ldots$; $m > 0$, $R(l) > 0$, $R(\beta-\gamma) > n-1$, (8)

and one of the following sets of conditions are satisfied :

(i) $\lambda > 0$, $|\arg. z| < \frac{1}{2}\lambda\pi$;

(ii) $\lambda \geqslant 0$, $|\arg. z| \leqslant \frac{1}{2}\lambda\pi$ and $R(\mu+1) \angle 0$.

Where λ, μ stand for $\sum_{j=1}^{q}(l_j) - \sum_{j=q+1}^{r}(l_j) + \sum_{i=1}^{p}(f_i) - \sum_{j=p+1}^{s}(f_1)$

and $\frac{1}{2}(r-s) + \sum_{j=1}^{s}(b_j) - \sum_{j=1}^{r}(a_j)$ respectively.

Proof: Substituting the value of H-function from (1) in the integrand of (8) we have

$$\int_0^1 x^{l-1}(1-x)^{\beta-\gamma-n}\,{}_2F_1(-n,\beta;\gamma;x)\left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{p}\Gamma(b_j-f_j\xi)}{\prod_{j=p+1}^{s}\Gamma(1-b_j+f_j\xi)}\times\frac{\prod_{j=1}^{2}\Gamma(1-a_j+l_j\xi)}{\prod_{j=q+1}^{r}\Gamma(a_j-l_j\xi)}\,z^{\xi}\,x^{m\xi}\,d\xi\right\}dx. \qquad \text{(A)}$$

Now we interchange the order of integration in (A) and evaluate the x-integral by virtue of (6) and obtain

$$\frac{\Gamma(\gamma)\,\Gamma(\beta-\gamma+1)}{\Gamma(\gamma+n)\,2\pi i}\int_L \frac{\prod_{j=1}^{p}\Gamma(b_j-f_j\xi)\prod_{j=1}^{q}\Gamma(1-a_j+l_j\xi)}{\prod_{j=p+1}^{s}\Gamma(1-b_j+f_j\xi)\prod_{j=q+1}^{r}\Gamma(a_j-l_j\xi)}$$
$$\times\frac{\Gamma(l+m\xi)\,\Gamma(\gamma+n-l-m\xi)}{\Gamma(\gamma-l-m\xi)\,\Gamma(\beta+l+1-\gamma+m\xi)}\,z^{\xi}\,d\xi \qquad \text{(B)}$$

Interpreting (B) with the help of (1) we get the required result.

The contour L runs from $-i\infty$ to $+i\infty$ such that all the poles of $\Gamma(b_j-f_j\xi)$, $(j=1,2,\ldots,p)$, $\Gamma(\gamma+n-l-m\xi)$ are to the right, and all the poles of $\Gamma(1-a_j+l_j\xi)$, $(j=1,2,\ldots,q)$, $\Gamma(l+m\xi)$ are to the left of the contour.

Regarding the interchange of the order of integration in (A) we see that x-integral is absolutely convergent if $R(l) > 0$, $R(\beta-\gamma) > n-1$ and $n=0, 1, 2\ldots$; To investigate the convergence of the ξ —integral in (A) we put $\xi = it$, $z = R_e{}^{i\phi}$ and then take the limit as t tends to infimity. Since [1; p. 47];

$$\Gamma(x+it) = (2\pi)^{\frac{1}{2}}\,|t|^{x-\frac{1}{2}}\,e^{-\frac{1}{2}\pi|t|} \qquad (x,\ t \text{ real})$$

$\text{Lim}_{t\to\infty}$

The absolute value of the integrand in the ξ—integral of (A) is comparable with the expression :

$$e^{-\frac{1}{2}\lambda\pi|t|}\ |t|^{\mu}\ e^{\phi t}$$

Where λ and μ have the same meaning as in (8) and $\phi = |\arg(z)|$. So ξ—integral in (A) is absolutely convergent in one of the following cases :—

(i) $\lambda > O$, $|\arg(z)| \angle \frac{1}{2}\lambda\pi$

(ii) $\lambda \geqslant O$, $|\arg(z)| \leqslant \frac{1}{2}\lambda\pi$ and $R(\mu+1) < O$.

The conditions of absolute convergence of the resulting integral (B) can be obtained by a method similar to that given above. Thus x—integral, ξ—integral and the resulting integral (B) are absolutely convergent under conditions stated with the theorem. Hence the interchange of the order of integration in (A) is justified by virtue of de la Vallee Poussin's theorem [7 ; p. 504].

4. *Second integral* :

$$\int_0^1 x^{l-1}(1-x)^{\beta-l-1}\ {}_2F_1(\alpha,\beta;\gamma;x)$$

$$\times H^{p,q}_{r,s}\left[z x^n (1-x)^{-n} \left| \begin{matrix} (a_1,l_1),\ldots,(a_r,l_r) \\ (b_1,f_1),\ldots,(b_s,f_s) \end{matrix} \right. \right] dx = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}$$

$$\times H^{p+2,q+1}_{r+2,s+2}\left[z \left| \begin{matrix} (1-l,n),(a_1,l_1),\ldots,(a_r,l_r),(\gamma-l,n) \\ (\beta-l,n),(\gamma-\alpha-l,n),(b_1,f_1),\ldots,(b_s,f_s) \end{matrix} \right. \right] \tag{9}$$

Where $n > O$, $R(l) > O$, $R(\beta-l) > O$, $R(\gamma-\alpha-l) > O$, and one of the following sets of conditions are satisfied :

(i) $\lambda > O$, $|\arg(z)| \angle \frac{1}{2}\lambda\pi$;

(ii) $\lambda \geqslant O$, $|\arg(z)| \leqslant \frac{1}{2}\lambda\pi$, $R(\mu+1) < O$ and $R(\mu+\beta-\alpha) < O$.

In (9), λ and μ have got the same meanings as in (8).

To prove (9) we proceed on the same lines as given during the proof of (8) except that here result (7) is used instead of (6).

5. *Particular cases.* On taking each e and f in (8) and (9) equal to σ, where σ is a positive integer, we respectively get the following integrals :

First Integral :

$$\int_0^1 x^{l-1}(1-x)^{\beta-\gamma-n}\ {}_2F_1(-n,\beta;\gamma;x)\ G^{p,q}_{r,s}\left[z^{1/\sigma} x^{m/\sigma} \left| \begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix} \right. \right] dx$$

$$= \frac{\Gamma(\gamma)\ \Gamma(\beta-\gamma+1)}{\Gamma(\gamma+n)}\ (2\pi)^{(p+q-\frac{1}{2}r-\frac{1}{2}s)(1-\sigma)}\ \sigma^{\sum_{j=1}^{s} b_j - \sum_{j=1}^{r}(a_j)+\frac{1}{2}r-\frac{1}{2}s+1}$$

$$\times m^{\gamma+n-\beta-1}$$

$$\times\, G^{m+\sigma p,\, m+\sigma q}_{2m+\sigma r,\, 2m+\sigma s}\left[z\,\sigma^{\sigma(r-s)} \,\middle|\, \begin{matrix} \Delta(m,1-l),\Delta(\sigma,a_1),\ldots, \\ \Delta(m,\gamma+n-l),\Delta(\sigma,b_1),\ldots, \end{matrix} \right.$$

$$\left. \begin{matrix} \Delta(\sigma,a_r),\ \Delta(m,\gamma-l) \\ \Delta(\sigma,b_s),\ \Delta(m,\gamma-\beta-l) \end{matrix} \right] \tag{10}$$

Where $n=0, 1, 2, \ldots$; $m>O$, $R(l)>O$, $R(\beta-\gamma)>n-1$, and one of the following sets of conditions are satisfied :

(i) $p+q-\frac{1}{2}s-\frac{1}{2}s>O$, $|\arg z| < (p+q-\frac{1}{2}r-\frac{1}{2}s)\,\sigma\,\pi$;

(ii) $p+q-\frac{1}{2}r-\frac{1}{2}s \geqslant O$, $|\arg z| \leqslant (p+q-\frac{1}{2}r-\frac{1}{2}s)\,\sigma\,\pi$ and

$$R\left(\sum_{j=1}^{r}(a_j)-\sum_{j=1}^{s}(b_j)+\tfrac{1}{2}s-\tfrac{1}{2}r-1\right)>O.$$

Second Integral :

$$\int_0^1 x^{l-1}(1-x)^{\beta-l-1}\,{}_2F_1(\alpha_1\,\beta;\gamma;x)\,G^{p,q}_{r,s}\left[z^{1/\sigma}x^{n/\sigma}(1-x)^{-n/\sigma}\,\middle|\,\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix}\right]dx$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}(2\pi)^{(p+q-\frac{1}{2}r-\frac{1}{2}s)(1-\sigma)+1-n}\;\sigma^{\sum_{j=1}^{s}(b_j)-\sum_{j=1}^{r}(a_j)+\frac{1}{2}r-\frac{1}{2}s+1}\;n^{\beta-\alpha-1}$$

$$\times\, G^{2n+\sigma p,\, n+\sigma q}_{2n+\sigma r,\, 2n+\sigma s}\left[z\sigma^{\sigma(r-s)} \,\middle|\, \begin{matrix} \Delta(n,1-l),\Delta(\sigma,a_1),\ldots,\Delta(\sigma,a_r),\Delta(n,\gamma-l) \\ \Delta(n,\beta-l),\Delta(n,\gamma-\alpha-l),\Delta(\sigma,b_1),\ldots,\Delta(\sigma,b_s)\end{matrix}\right] \tag{11}$$

Where n is a positive integer, $R(l)>O$, $R(\beta-l)>O$, $R(\gamma-\alpha-l)>O$, and one of the following sets of conditions are satisfied :

(i) $p+q-\frac{1}{2}r-\frac{1}{2}s>O$, $|\arg z| < (p+q-\frac{1}{2}r-\frac{1}{2}s)\,\sigma\,\pi$;

(ii) $p+q-\frac{1}{2}r-\frac{1}{2}s \geqslant O$, $|\arg z| \leqslant (p+q-\frac{1}{2}r-\frac{1}{2}s)\,\sigma\,\pi$,

$$R\left(\sum_{j=1}^{r}(a_j)-\sum_{j=1}^{s}(b_j)+\tfrac{1}{2}s-\tfrac{1}{2}r-1\right)>O \quad \text{and}$$

$$R\left(\sum_{j=1}^{r}(a_j)-\sum_{j=1}^{s}(b_j)+\tfrac{1}{2}s-\tfrac{1}{2}r+\alpha-\beta\right)>O.$$

If $\sigma=1$ in (10) and (11) we obtain both the results given by Sharma [5; p. 539].

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INVESTIGATION OF 1022 DOUBLES IN THE OXFORD ASTROGRAPHIC CATALOGUES +27° to +29° WITH AN ANGULAR SEPARATION LESS THAN 15″ (Part V)

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ABSTRACT

The present paper is a continuation of several previous papers (1,2,3,4,5,6) and gives the results of the search and counts of 1022 doubles with an angular separation less than 15″ in the Oxford Astrographic Catalogues + 27° to + 29° in different magnitudes for different values of $\triangle$m (the difference of magnitude between the components). The average galactic concentration obtained by comparing the distributions in galactic latitudes $0° < | \beta | \leqslant 20°$ and $| \beta | > 40°$ works out to be 4.310 which is smaller than that for the zones +32° and +33° (7.77), +26° and 25° (6.09). The average galactic concentration for all stars in these zones is found to lie between 2.24 and 3.07. The average galactic concentration for doubles in these zones is larger than the average galactic concentration of stars themselves, a conclusion previously arrived at. The method of investigation remains the same as described in the first paper. The ratio of the observed to optical (T:O) can be easily computed from the data given in the tables I to IX.

Observed distribution of stars according to *d*, *m*, $\triangle$ m.

Table I +27° ; 0″ < $d \leqslant 5″$.

$\triangle m$ / m	.3	.6	.9	1.2	1.5	1.8	2.1	2.4	>2.4	T	O
< 9.0	0	0	1	0	0	1	0	0	1	3	0.9
10.0	2	0	1	0	0	1	1	1	0	6	0.7
11.0	2	3	4	2	2	2	1	0	0	16	3.7
12.0	11	5	4	2	1	0	0	0	0	23	3.1
13.0	4									4	0.9

Table II +27° ; 5″ < $d \leqslant 10″$.

$\triangle m$ / m	.3	.6	.9	1.2	1.5	1.8	2.1	2.4	>2.4	T	O
< 9.0	2	0	1	1	2	1	1	1	6	15	8.4
10.0	1	3	2	5	0	3	2	1	3	20	6.3
11.0	10	10	1	3	2	3	3	0	1	33	33.2
12.0	16	15	8	0	2	0	1	1	0	43	28.3
13.0	12	2	2	2						18	8.1
>	2									2	..

Table III +27° ; 10″ < $d \leqslant 15″$.

$\triangle m$ / m	.3	.6	.9	1.2	1.5	1.8	2.1	2.4	>2.4	T	O
< 9.0	0	1	1	0	2	0	1	0	7	12	25.2
10.0	0	2	3	1	2	3	2	1	3	17	18.8
11.0	7	6	9	3	3	6	2	0	1	37	99.6
12.0	27	8	10	1	2	1	2	1	0	52	84.8
13.0	9	5	2	1						17	24.2
>	2	2	1								

Table IV $+28^0$; O $< d \leqslant 5''$.

$\frac{\Delta m}{m}$	.3	.6	.9	1.2	1.5	1.8	2.1	2.4	>2.4	T	O
< 9.0	0	1	0	1	0	0	1	0	0	3	0.9
10.0	0	3	0	1	4	0	0	0	0	8	0.8
11.0	4	8	2	4	2	0	0	0	0	20	3.9
12.0	10	11								21	3.0
13.0	1	3								4	1.0
>	0	1								1	..

Table V $+28^0$; $5'' < d \leqslant 10''$

< 9.0	1	0	1	0	0	1	0	2	6	11	8.9
10.0	5	0	1	4	5	4	1	1	0	21	7.5
11.0	11	10	11	8	2	4	1	1	0	48	34.9
12.0	12	6	3	2	2					43	26.8
13.0	15	3	3	0	1					22	9.0
>	2									2	..

Table VI $+28^\circ$; $10'' < d \leqslant 15''$.

< 9.0	1	0	0	2	0	0	1	0	6	10	26.7
10.0	1	0	3	3	2	2	4	0	2	17	22.5
11.0	12	10	7	7	4	4	0	0	2	46	104.7
12.0	10	9	10	3	3	2	2	0	0	39	80.4
13.0	7	3	3							13	26.9

Table VII $+29^\circ$; $O'' < d \leqslant 5''$.

< 9.0	1	1	2	0	0	0	0	0	1	5	0.1
10.0	0	1	0	0	1	1	0	1	0	4	0.9
11.0	6	5	6	2	1	1	0	0	0	21	4.2
12.0	8	8	1	0	1	0	0	0	0	18	3.9
13.0	7	3								10	0.9

Table VIII $+29^\circ$; $5'' < d \leqslant 10''$

< 9.0	0	3	0	1	3	2	2	3	7	21	1.0
10.0	3	2	4	1	0	5	1	2	2	20	8.5
11.0	8	9	9	11	6	3	0	1	0	47	37.7
12.0	16	9	4	8						37	34.7
13.0	7	4	4	1						16	8.6

Table IX $+29^0$; $10'' < d \leqslant 15''$.

< 9.0	0	0	1	3	2	3	0	1	14	24	3.1
10.0	1	1	1	2	1	6	1	3	5	21	25.6
11.0	3	10	11	9	7	3	1	0	0	44	113.0
12.0	28	15	13	2	0	1				59	104.1
13.0	15	7	1							23	25.8

Totals :

Table										
I	19	8	10	4	3	4	2	1	1	52
II	43	30	14	11	6	7	7	3	10	131
III	45	24	26	6	9	10	7	2	11	140
IV	15	27	2	6	6	0	1	0	0	57
V	52	25	22	15	10	11	2	4	6	147
VI	31	22	23	15	9	8	7	0	10	125
VII	22	18	9	2	3	2	0	1	1	58
VIII	34	27	21	22	9	10	3	6	9	141
IX	47	33	27	16	10	13	2	4	9	171

Observed distribution of stars according to d, m, $\triangle m$, β.

Table X ($+27°$ to $+29°$) ; $0'' < d \leqslant 5''$.

$0° < |\beta| \leqslant 20°$.

Δm / m	.3	.6	.9	1.2	1.5	1.8	2.1	2.4	> 2.4	T
< 9.0	0	0	2	1	0	1	1	0	1	6
10.0	1	1	1	1	4	0	1	0	0	9
11.0	9	9	10	4	2	2	1	0	0	37
12.0	16	11	2	1	1					31
13.0	5	2								7

Table XI $0'' < d \leqslant 5''$

$20° < |\beta| \leqslant 40°$.

Δm / m	.3	.6	.9	1.2	1.5	1.8	2.1	2.4	> 2.4	T
< 9.0	1	1	1	0	0	0	0	0	0	3
10.0	0	2	0	0	1	2	0	1	0	6
11.0	2	2	1	3	1	0	0	0	0	9
12.0	9	10	2	1	0	0	0	0	0	22
13.0	6	2								8
>	0	1								

Table XII $0'' < d \leqslant 5''$

$|\beta| > 40°$

Δm / m	.3	.6	.9	1.2	1.5	1.8	2.1	2.4	> 2.4	T
< 9.0	0	1	0	0	0	0	0	0	1	2
10.0	1	1	0	0	0	0	0	1	0	3
11.0	1	5	1	1	2	1	0	0	0	11
12.0	4	3	1	0	1	0	0	0	0	9
13.0	1	2								3

Table XIII 5″ $d \leqslant 10''$

$0^\circ < |\beta| \leqslant 20^\circ$

$\frac{\Delta m}{m}$	.3	.6	.9	1.2	1.5	1.8	2.1	2.4	> 2.4	T
< 9.0	1	0	0	1	4	2	1	4	9	22
10.0	3	1	5	7	3	5	2	2	1	29
11.0	16	20	11	14	6	3	3	1	1	75
12.0	32	15	13	8	4	1	0	1	0	74
13.0	18	3	4	1	1					27
>	2									2

Table XIV 5″ $< d \leqslant 10''$

$20^\circ < |\beta| \leqslant 40^\circ$

< 9.0	0	3	1	1	1	1	1	2	9	19
10.0	4	2	2	2	1	4	1	1	2	19
11.0	10	6	9	7	4	5	0	1	0	42
12.0	9	19	4	1	0	0	1	0	0	34
13.0	10	5	4	2						21
	2									

Table XV 5″ $< d \leqslant 10''$

$|\beta| > 40^\circ$

< 9.0	2	0	1	0	0	1	1	0	1	6
10.0	2	2	0	1	1	3	1	1	2	13
11.0	3	5	1	1	0	2	1	0	0	11
12.0	9	2	1	2	0	1	0	0	0	15
13.0	6	1	1							8

Table XVI 10″ $< d \leqslant 15''$

$0^\circ < |\beta| \leqslant 20^\circ$

< 9.0	0	0	2	2	1	0	1	0	18	24
10.0	2	2	5	3	3	4	4	2	7	32
11.0	17	21	15	16	10	10	2	0	3	94
12.0	40	24	21	4	1	4	3	1	0	98
13.0	16	10	5							31
>	0	2	1							3

Table XVII $10'' < d \leqslant 15''$

$20^\circ < |\beta| \leqslant 40^\circ$

$\frac{\triangle m}{m}$	.3	.6	.9	1.2	1.5	1.8	2.1	2.4	> 2.4	T
< 9.0	1	1	0	3	1	1	1	1	7	16
10.0	0	1	2	3	1	4	2	0	3	16
11.0	3	4	9	3	3	2	1	0	0	25
12.0	19	5	9	2	3	0	1	0	0	39
13.0	14	5	1	1						21
	2									2

Table XVIII $10'' < d \leqslant 15''$

$|\beta| < 40^\circ$

< 9.0	0	0	0	0	2	2	0	0	2	6
10.0	0	0	0	0	1	3	1	2	0	7
11.0	2	1	3	0	1	1	0	0	0	8
12.0	6	3	3	0	1	0	0	0	0	13
13.0	1									1

Totals Tables

X	31	23	15	7	7	3	3	0	1	90
XI	18	18	4	4	2	2	0	1	0	49
XII	7	12	2	1	3	1	0	1	1	28
XIII	82	39	33	31	18	11	6	8	11	229
XIV	35	35	20	13	6	10	3	4	11	137
XV	22	8	4	4	1	7	3	1	3	53
XVI	75	59	49	25	15	18	10	3	28	282
XVII	39	16	21	12	8	7	5	1	10	119
XVIII	9	4	6	0	5	6	1	2	2	35

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CARBON, NITROGEN AND SULPHUR STATUS OF SOME ALKALI AND ADJOINING SOIL PROFILES OF UTTAR PRADESH

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ABSTRACT

Three alkali and their adjoining soil profile samples were analysed for organic carbon, total nitrogen and total sulphur contents. The average values of these elements were found to be 0.3804, 0.045 and 0.0323 percent respectively in alkali soil profiles as compared with 0.6277, 0.060 and 0.0275 percent in adjoining soil profiles respectively.

The averege ratios between C and N; and C and S, are found to be 11.2 and 14.5 in alkali soil profiles respectively as compared with 11.5 and 31.4 in adjoining soil profiles. The ratios C: N: S are found to be 112 : 10 : 9.5 and 115 : 10 : 4.5 in alkali and adjoining soil profiles respectively.

Carbon, nitrogen and sulphur play an important role in the build up of soil organic matter. The relationship between carbon and nitrogen has been recognised for a long time. Truchot (1875) was the first to give the idea of carbon and nitrogen relationship in the soil. As early as 1883, Lewis *et. al,* recognised the significance of C/N ratio and its quantitative use for diagnosing soil and subsoil characteristics. Williams *et. al.* (1960) have recognised the close association of carbon, nitrogen and sulphur with one another in soil organic matter.

In the present investigation, a depthwise distribution of carbon, nitrogen and sulphur has been studied to see the relationship between them in some soils representing alkali and adjoining soil profiles of Uttar Pradesh.

EXPERIMENTAL

Samples from three alkali soil profiles as well as from their adjoining cultivable fields were collected from districts of Uttar Pradesh, *viz.*, Ballia, Jaunpur and Varanasi. Genetically all the soils are Gangetic alluvium affected by salinity and alkalinity. All the soil samples were found to be on the alkaline side with pH ranging from 7.2 to 9.4.

Organic carbon was determined by Walkley and Black's method; total nitrogen by Kjeldahl method; and calcium carbonate by rapid titration method following those described in Piper (1950). Total sulphur was determined by sodium carbonate and sodium nitrate fusion and precipitation as barium sulphate (Robinson, 1945).

RESULTS AND DISCUSSION

The results of chemical analysis are presented in table 1. An examination of the data for total nitrogen reveals no definite trend of its distribution in all the alkali and adjoining soil profiles except the marked lower values in the (0-2") surface layers of alkali soil profiles. This shows no signs of nitrogen leaching in the soil. Total nitrogen content ranges from 0.021 to 0.051% with an average of 0.034% and from 0.035 to 0.090% with an average of 0.06% in the alkali and adjoining soil profiles respectively. The higher values of total nitrogen in adjoining soil profiles may be due to cultivation practices.

Similar to the distribution pattern of total nitrogen, total organic carbon also did not show any regular order of distribution in the profiles. Total organic carbon ranges from 0.1827 to 0.7048% with an average of 0.3804% and from 0.4758 to 0.8422% with an average of 0.6277% in alkali and adjoining soil profiles respectively. Again, the higher values of organic carbon are found in adjoining soil profiles than in alkali soil profiles. Values of both organic carbon and total nitrogen showing a nearly similar trend down the profiles indicate a close relationship between the two elements.

Further, total sulphur content also shows no definite order of distribution in the profiles. Total sulphur content ranges from 0.0121 to 0.0740% with an average of 0.0323% and from 0.0137 to 0.0589% with an average of 0.0275% in alkali and adjoining soil profiles respectively. Contrary to carbon and nitrogen, sulphur content of alkali soil profiles are found higher than those in adjoining soil profiles.

Calcium Carbonate and C, N, and S.

Accumulation of calcium carbonate is found in both types of profiles of Ballia and Jaunpur, while profiles from Varanasi do not show such accumulation (Table 1).

TABLE 1

Carbon, Nitrogen, and Sulphur contents of the Alkali and Adjoining soils

Depth in inches	Calcium carbonate %	Texture	Total carbon %	Total Nitrogen %	Total Sulphur %
		1. Abhanpur (Ballia)-Alkali soil			
0-2	0.2	Clay loam	0.2114	0.021	0.0322
2-10	0.3	Sandy loam	0.4680	0.042	0.0562
10-20	0.7	,, ,,	0.2708	0.038	0.0493
20-35	1.5	Clay loam	0.2028	0.038	0.0288
35-50	1.4	Clayey	0.3588	0.032	0.0356
50-65	15.0	Loam	0.6318	0.039	0.0740
65-74	22.0	Clay loam	0.7048	0.038	0.0438
		2. Abhanpur (Ballia)-Adjoining Soil			
0-8	1.3	Sandy loam	0.8422	0.087	0.0438
8-20	16.6	,, ,,	0.5148	0.089	0.0617
20-32	19.0	,, ,,	0.6942	0.090	0.0562
32-48	25.2	,, ,,	0.4758	0.081	0.0589
48-65	16.0	Loam	0.7322	0.083	0.0260
65-72	26.5	Clay loam	0.6318	0.085	0.0370

TABLE 1.—(Contd.)

Depth in inches	Calcium carbonate %	Texture	Total carbon %	Total Nitrogen %	Total Sulphur %
		3. Lagdharpur (Jaunpur)-Alkali soil			
0- 2	0 2	Clay loam	0.1817	0.022	0.0223
2- 8	0.2	Sandy loam	0.4124	0.040	0.0288
8-20	0.8	Loam	0.4524	0.044	0.0387
20-32	9.0	Clay loam	0.5148	0.030	0.0480
32-44	7.1	,, ,,	0.5382	0.051	0.0493
44-62	1.0	,, ,,	0.4290	0.029	0.0356
		4. Lagdharpur (Jaunpur)-Adjoining soil			
0- 6	0.8	Loam	0.7878	0.063	0.0178
6-19	Nil	Clayey	0.6318	0.043	0.0137
19-35	5.4	Silt loam	0.6318	0.054	0.0192
35-52	7.9	Clayey	0.5928	0.051	0.0223
52-65	12.5	Clay loam	0.5928	0.047	0.0192
		5. Korajpur (Varanasi)-Alkali soil			
0-2	0.3	Clay loam	0.1916	0.021	0.0121
2-12	Nil	Clayey	0.3424	0.039	0.0192
12-24	Nil	Clay loam	0.3562	0.027	0.0123
24-36	Nil	Clayey	0.3580	0.042	0.0096
36-50	Nil	,,	0.3502	0.041	0.0137
50-66	0.4	,,	0.3424	0.034	0.0178
66-75	2.5	Clay loam	0.2956	0.021	0.0192
		6. Korajpur (Varanasi)-Adjoining soil			
0-10	Nil	Sandy loam	0.6084	0.043	0.0178
10-24	Nil	Clay loam	0.5616	0.049	0.0178
24-36	Nil	Silt loam	0.6464	0.039	0.0137
36-48	Nil	Clayey	0.4992	0.046	0.0192
48-60	Nil	,,	0.6786	0.037	0.0082
60-70	0.3	,,	0.5488	0.035	0.0137

It shows a positive regular rise in depth with a few exceptions, implying a build up of calcium carbonate with depth to be apparently an epigenic process. Average of organic carbon, total nitrogen and total sulphur with calcium carbonate are given in table 2. It will be seen that organic carbon content bears an apparent positive relationship ($r = +0.82$) with calcium carbonate in alkali soil profiles, but no apparent relationship ($r = +0.20$) is seen in adjoining soil profiles. The total nitrogen content shows no apparent relationship ($r = +0.18$) with calcium carbonate in alkali soil profiles but a significant positive correlation ($r = +0.73$) is seen in adjoining soil profiles. Further, total sulphur shows significant positive correlation ($r = +0.59$) and ($r = +0.74$) both at 1% level in alkali and adjoining soil profiles respectively.

TABLE 2

Carbon, Nitrogen and Sulphur affected by Calcium Carbonate
(Average in percent)

Calcium Carbonate range %	Organic Carbon		Total Nitrogen		Total Sulphur	
	Alkali	Cultivated	Alkali	Cultivated	Alkali	Cultivated
0- 1	0.3327	0.6203	0.033	0.045	0.0262	0.0152
1-2	0.3302	0.8422	0.033	0.087	0.0333	0.0433
2-10	0.5265	0.6123	0.041	0.053	0.0324	0.0213
10-20	0.6318	0.6335	0.039	0.077	0.0740	0.0408
> 20	0.7048	0.5538	0.036	0.083	0.0438	0.0480

Texture and C, N, and S.

The texturewise averages of carbon, nitrogen and sulphur are given in table 3. An examination of this reveals no apparent relationship between organic carbon and fineness in texture in alkali ($r = -0.07$) and adjoining soil ($r = -0.14$) profiles. The total nitrogen content shows no significant correlation ($r = -0.24$) with texture in alkali soil profiles, while a significant negative correlation ($r = -0.80$) is found in adjoining soil profiles between the two factors. Further, the total sulphur reveals a negative relationship with texture in alkali ($r = -0.57$) and adjoining soil ($r = -0.78$) profiles respectively.

TABLE 3

Carbon, Nitrogen and Sulphur affected by Texture
(Average as percent)

Texture	Alkali soil profiles			Adjoining soil profiles		
	Organic Carbon	Total N	Total S	Organic Carbon	Total N	Total S
Sandy loam	0.3337	0.040	0.0448	0.6271	0.078	0.0477
Loam	0.5421	0.042	0.0564	0.7600	0.076	0.0219
Silt loam	—	—	—	0.6391	0.047	0.0165
Clay loam	0.3621	0.030	0.0304	0.5954	0.060	0.0133
Clayey	0.3504	0.038	0.0192	0.5902	0.042	0.0153

Carbon and nitrogen

Stevenson (1959) stated that the C/N ratio of surface soils of agricultural importance in temperate regions has a frequency of 10 to 12. Satyanarayana *et. al.* (1946) reported the same frequency for black soils of semi-arid regions but in a comparative study of Indian soils he found the ratio to fluctuate widely from 5 to 25. In the present investigation instances are found in all the alkali and adjoining soil profiles individually (Table 4) where the C/N ratio narrows with depth.

TABLE 4

Derived data for soil samples of the Alkali and Adjoining soil profiles

Profile No.	Depth in inches	C/N ratio	C/S ratio	C : N : S ratio
1	0-2	10.1	6.6	101 : 10 : 15
	2-10	11.1	8.3	111 : 10 : 13
	10-20	7.1	5.5	71 : 10 : 13
	20- 35	5.1	7.0	53 : 10 : 8
	35- 50	11.2	10.1	112 : 10 : 11
	50- 65	16.2	8.5	162 : 10 : 19
	65- 75	19.5	16.1	195 : 10 : 12
2	0- 8	5.8	19.2	58 : 10 : 5
	8-20	7.7	8.3	77 : 10 : 7
	20-32	5.8	12.3	59 : 10 : 6
	32-48	8.8	8.1	88 : 10 : 7
	48-65	7.4	28.0	74 : 10 : 3
	65-72	10.3	17.1	103 : 10 : 4
3	0-2	8.3	8.1	83 : 10 : 10
	2-8	10.3	14.3	103 : 10 : 7
	8-20	10.3	11.7	103 : 10 : 9
	20-32	17.1	10.7	171 : 10 : 16
	32-44	10.6	10.9	106 : 10 : 10
	44-62	14.8	12.1	148 : 10 : 12
4	0-6	11.6	44.3	116 : 10 : 3
	6-19	14.7	46.1	147 : 10 : 3
	19-35	11.7	32.9	117 : 10 : 4
	35-52	11.6	25.4	116 : 10 : 5
	52-65	12.6	30.8	126 : 10 : 4
5	0-2	9.1	15.8	91 : 10 : 6
	2-12	8.8	17.8	88 : 10 : 5
	12-24	13.0	28.5	130 : 10 : 5
	24-36	8.5	37.3	85 : 10 : 2
	36-50	8.5	25.6	85 : 10 : 3
	50-60	10.1	19.2	101 : 10 : 5
	60-75	14.1	15.4	141 : 10 : 9
6	0-10	14.1	34.2	141 : 10 : 4
	10-24	11.4	31.6	114 : 10 : 4
	24-36	16.6	47.2	166 : 10 : 4
	36-48	10.9	26.0	108 : 10 : 4
	48-60	18.3	82.8	183 : 10 : 2
	60-70	15.7	40.1	157 : 10 : 4

Instances are also found where the ratio widens with depth in the profiles. This shows a relative enrichment of nitrogen and carbon in different horizons of the profiles. Profile 4 is the only profile when the ratio ranges very narrowly (from 11.6 to 12.6). All the other profiles show a wider range for the ratio. Narrowing of the ratio in profile 4 supports the decomposition of organic matter in the profile (Russell, 1950). Statistical analysis of the data for the two factors shows a significant positve correlation ($r = + 0.51$) in alkali soil profiles and no significant correlation ($r = + 0.27$) in adjoining soil profiles.

Carbon and Sulphur

Williams *et. al.* (1960) reported that sulphur was highly correlated with carbon and nitrogen in all groups of soils he studied. From table 4, a wider variation in C/S ratio is found in all the profiles individually with exceptionally wider variations in profile 6. Statistical analysis of the data for the two factors shows a significant positive correlation ($r = + 0.64$) at 1% level in alkali soil profiles and no significant correlation ($r = + 0.01$) in adjoining soil profiles.

Carbon, Nitrogen and Sulphur

Williams *et. al.* (1960) reported an average C: N: S ratio for all groups of soils studied by him to be 140: 10: 1.4. Williams and Donald (1957) reported nearly a similar ratio C: N: S = 155: 10: 1.4; Walker and Adams (1958) a ratio C: N: S = 129: 10: 1.3 and Evans and Rost (1945) a ratio C: N: S = 122 : 10: 1.5. The C: N: S ratio of all the soil samples studied in the present investigation are given in table 4 and the average ratios for alkali and adjoining soil profiles are 112.10: 9.5 and 115: 10: 4.5. The results obtained show that the sulphur contents of the soil under present study are much higher than those reported by the above investigators.

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CERIMETRIC ESTIMATION OF THIOUREA, POTASSIUM THIOCYANATE, POTASSIUM FERROCYANIDE AND POTASSIUM FERRICYANIDE

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ABSTRACT

In 50% sulphuric acid, thiourea, thiocyanate, ferrocyanide and ferricyanide have been oxidized by ceric sulphate requiring 8, 6, 13, and 12 equivalents, respectively. In case of ferrocyanide and ferricyanide, ferric alum has been used as a fixer of cyanide group.

INTRODUCTION

Ferrocyanide requires one equivalent in titration against ceric sulphate in presence of moderately concentrated sulphuric acid and hydrochloric acid[1-7]. Six equivalents of ceric sulphate are consumed in oxidation of potassium thiocyanate in presence of 6-N or more sulphuric acid[8]. Oxidation of thiourea and potassium ferricyanide appear to have not been attempted. We have in our investigation attempted the oxidation of these substances in presence of high acid concentration.

EXPERIMENTAL

All the reagents except thiourea and potassium thiocyanate were of A. R. grade. These two substances were used after recrystallisation and drying. Water redistilled over alkaline permanganate was used throughout the investigation. 5 ml of 0.20M ceric sulphate solution in 2.5 N sulphuric acid and 5 ml of strong sulphuric acid were taken in a clean and dry 250 ml conical flask in which a ground glass air condenser could fit in. Redistilled water or 1 ml of saturated ferric alum or 3 ml of saturated silver sulphate solutions were added to it where necessary and then ice cooled if necessary. The reductant solutions of urea, thiourea, thiocyanate, ferrocyanide or ferricyanide were added to the mixture. Ground glass joint air condenser was fitted in and then the flask was heated over a low voltage heater, till fumes of sulphuric acid just evolved. It was then cooled. The volume in the conical flask was made upto nearly 100 ml with redistilled water. It was subsequently heated to dissolve the unreacted separated ceric sulphate complex, cooled and the remaining Ce (iv) was back titrated against 0.05M ferrous ammonium suphate solution using ferroin as indicator. The blank was also run.

RESULTS AND DISCUSSION

5 ml each of 0.20M ceric sulpate and sulphuric acid (A.R., B.D.H.) were used.

Substance	Saturated ferric alum (A. R.) soln.	Saturated silver sulphate soln.	Procedure	Equivalent required
1. 1 ml 0.05M urea solution	—	—	heated directly on plate heater	nil
2. 1 ml 0.05M thiourea solution	—	—	,,	7.98, 7.96
3. (i) 1 ml 0.05M KCNS solution (standardised by Volhard's method)	—	—	diluted Ce (vi)-H_2SO_4 mixture to about 100 ml with water and added to KCNS soln and heated.	4.51, 5.02

Substance	Saturated ferric alum (A.R.) soln.	Saturated silver sulphate soln.	Procedure	Equivalents required.
(ii) 1 ml 0.05M KCNS solution (standerdised by Volhard's method)	1 ml	—	heated directly without dilution with water.	6.03, 6.02
(iii) ”	1 ml	—	Ce (iv)-H_2SO_4 mix. was ice cooled. The KCNS solution was added to it slowly with the help of a pipette and then heated on the plate heater.	6.12, 6.00
(iv) ”	—	3 ml	Ce (iv)-H_2SO_4-silver sulphate solution was ice cooled. KCNS solution was added slowly to it and then heated.	6.13, 6.21
4. (i) 1 ml 0.03M $K_4Fe(CN)_6$	—	—	heated directly	2.31, 3.02
(ii) ”	—	—	$K_4Fe(CN)_6$ was slowly added to ice cooled Ce (iv) - H_2SO_4 mix. and then heated	6.07, 6.31
(iii) ”	1 ml	—	K_4 $Fe(CN)_6$ solution was slowly added to ice cooled Ce (iv) - H_2SO_4 - F (III) mix. and then heated.	13.03, 13.06
(iv) ”	—	3 ml	K_4 $Fe(CN)_6$ solution was slowly added to ice cooled Ce(iv) - H_2SO_4-silver sulphate mix. and then heated	6.52, 6.61
5. (i) 1 ml 0.028M $K_3Fe(CN)_6$	—	—	heated directly	2.40, 3.23
(ii) ”	1 ml	—	K_3 $Fe(CN)_6$ solution was slowly added to the ice cooled Ce(iv) H_2SO_4-Fe (III) mix. and then heated	11.95, 12.02

A perusal of the table shows that urea is not oxidized at the cost of Ce(iv) which is accountable by the following reactions :

$$NH_2CO\,NH_2 \xrightarrow{\text{hydrolysis}} 2\,NH_3 + CO_2 \quad \text{(i)}$$

$$NH_2CO\,NH_2 \xrightarrow{\text{heat}} NH_4CNO \xrightarrow{\text{acid}} HCNO \xrightarrow{\text{hydrolysis}} NH_3 + CO_2 \quad \text{(ii)}$$

Thiourea on the other hand is oxidised at the cost of Ce(iv) requiring 8 equivalents. This can be explained by the following reaction.

$$NH_2CS\,NH_2 + 4\,O \longrightarrow NH_2CONH_2 + SO_3 \quad \text{(iii)}$$

$$NH_2CO\,NH_2 + H_2O \longrightarrow 2\,NH_3 + CO_2 \quad \text{(iv)}$$

This mechanism is quite in conformity with that proposed for the oxidation of thiourea by permanganate in sulphuric acid medium wherein urea is a product. The following reaction may also explain the requirement of 8 equivalents :

$$NH_2CS\,NH_2 + H_2O \longrightarrow 2\,NH_3 + CSO \quad \text{(v)}$$

$$CSO + 4\,O \longrightarrow CO_2 + SO_3 \quad \text{(vi)}$$

$$NH_2CS\,NH_2 \xrightarrow[\text{in presence of dil. sulphuric acid.}]{110^\circ\,C} NH_4CNS \xrightarrow{H_2SO_4} HCNS \quad \text{(vii)}$$

$$HCNS + 4\,O \longrightarrow HCNO + SO_3 \quad \text{(viii)}$$

The above proposed mechanisms (V, VI & vii, viii) are not in agreement with the results obtained in oxidation of KCNS. From the table it is seen that KCNS requires 6 equivalents instead of 8 as proposed by the mechanism (vii, viii). Also as HCNS readily hydrolyses in presence of sulphuric acid,[9] the equivalents of KCNS in oxidation should exceed 6 which is not supported by the observation Hence the mechanism (v, vi) proposed for the oxidation of thiourea is also ruled out.

The cerimetric oxidation of KCNS in sulphuric acid (more than 6-N) requiring 6 equivalents was proposed by Joshi and Deshmukh[8]

$$HCNS + 3\,O \longrightarrow HCN + SO_3$$

Our result indicates the above reaction taking place even in 50% sulphuric acid. The oxidation reaction is similar to the oxidation of KCNS with $KMnO_4$[10,11] and H_2O_2[12]. Results of Joshi and Deshmukh (Loc : cit.) show appreciable deviation in equivalence with increasing dilution of sulphuric acid. This is also evident from our observation (3i) in which acid strenth was 2N. The above investigators did not give any reason for it. We are of the opinion that in dilute acid the direct oxidation becomes slow and the thiocyanic acid hydrolyses to some extent as follows :

$$2\,HCNS + 2\,H_2O \longrightarrow 2\,NH_3 + CS_2 + CO_2$$

The carbon disulphide being volatile escapes oxidation leading to a fall in equivalence value.

Potassium ferrocyanide, which has been reported to take up one equivalent, in our investigation takes up 13 equivalents in presence of ferric alum, the acid concentration being nearly 9M.

Potassium ferricyanide takes up 12 equivalents under the same conditions. This shows that the complex $Fe_2(CN)_6$ strongly binds cyanide groups even in presence of 9M sulphuric acid and then on subsequent heating they are quantitatively oxidised to cyanate state.

$$CN^- + O \rightarrow CNO^-$$

In absence of ferric alum and even in presence of Ag ions part of cyanide groups escapes exidation.

CONCLUSIONS

From the investigation it is evident that cerimetric method can be employed for estimating thiourea, ferrocyanide and ferricyanide in presence of 50% sulphuric acid. In the latter two cases ferric alum should be used as binder of cyanide group and its mixture with $Ce(iv)-H_2SO_4$ should be ice cooled, ferro or ferricyanide solution slowly added to it and then heated gradually. Thiourea requires 8 equivalents, ferrocyanide 13 and ferricyanide 12 equivalents of Ce(iv).

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ON SCHWARZ DIFFERENTIABILITY—I

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ABSTRACT

In this paper some theorems analogous to Rolle's Theorem and Mean Value Theorems have been proved for Schwarz derivative. Also, a connection between the ordinary derivative and Schwarz derivative has been obtained with the help of the notion of uniform Schwarz differentiability.

INTRODUCTION

1. Let $f(x)$ be a real function of the real variable x defined in an open interval I of which $[a,b]=a \leqslant x \leqslant b$ is a closed subset. For $x \in$ I, if the limit

$$\lim_{h \to 0} \frac{f(x+h)-f(x-h)}{2h}$$

exists, it is called the Schwarz derivative [1] of $f(x)$ at the point x and is denoted by $f^{(1)}(x)$. It is known that if the ordinary derivative $f'(x)$ exists at the point x, then $f^{(1)}(x)$ also exists and $f'(x)=f^{(1)}(x)$. The converse, however, is not true. It is now natural to ask whether the main properties of the ordinary derivatives are also possessed by Schwarz derivatives. Indeed, it has been shown [4] that if $F(x)$ is continuous in $[a, b]$ and the second Schwarz derivative $F^{(11)}(x)=0$ everywhere on (a,b) then $F(x)$ is a linear function. Some sufficient conditions in terms of the Schwarz derivatives also have been obtained [4] in order that a function may be convex downwards. But it appears that nothing is known so far about Rolle's theorem, Mean value theorem and Darboux theorem for Schwarz derivatives. The present paper is an outcome of this concideration. The paper contains two sections, sec. A and sec. B. In sec. A we have attempted to prove the above theorems. The conditions, we require to prove the theorems, are more stringent than those required to prove the parallel theorems for ordinary derivatives. But this may not be unnatural if we have in mind that the existence of the Schwarz derivative at a point does not necessarily imply the existence of the ordinary derivative there.

In sec. B, we have introduced the notion of the point of uniform and non-uniform Schwarz differentiability and in this connection we have defined a function $d(x)$ in $[a,b]$ analogous to the function introduced by Lahiri [3] in connection with the point of uniform and non-uniform differentability of a function $f(x)$. We have established certain properties of $d(x)$ along with some other results on Schwarz differentiability which show how the existence of the ordinary derivative may be ensured from that of the Schwarz derivative. We shall use the notations of Hans Rademacher [5] for four derivatives of $f(x)$ namely the upper right, upper left, lower right and lower left derivatives of $f(x)$ will be denoted by $\overline{D}_r f(x)$, $\overline{D}_l f(x)$, $\underline{D}_r f(x)$ and $\underline{D}_l f(x)$, respectively.

2. *Theorem-1.* If $f(x)$ is continuous in $[a,b]$ and Schwarz differentiable at $\xi \in (a,b) = a < x < b$, then

$$(a)\ \tfrac{1}{2}\ \overline{D}_r f(\xi) \leqslant f^{(1)}(\xi) \leqslant \tfrac{1}{2}\ \underline{D}_l f(\xi)$$

$$\text{or } (b)\ \tfrac{1}{2}\ \overline{D}_l f(\xi) \leqslant f^{(1)}(\xi) \leqslant \tfrac{1}{2}\ D_r f(\xi)$$

according as $f(\xi)$ is the upper bound or lower bound of $f(x)$ in $[a,b]$

Proof. We prove the theorem for the case (a). The proof for the case (b) is analogous.

We have

$$\Delta(f, x, h) = \frac{f(x+h) - f(x-h)}{2h} = \frac{[f(x+h) - f(x)] + [f(x) - f(x-h)]}{2h}.$$

Since $f(\xi)$ is the upper bound, we have

$$\frac{f(\xi+h) - f(\xi)}{2h} \leqslant \Delta(f, \xi, h) \leqslant \frac{f(\xi-h) - f(\xi)}{-2h}$$

Taking limit we obtain

$$\tfrac{1}{2}\ \overline{D}_r f(\xi) \leqslant f^{(1)}(\xi) \leqslant \tfrac{1}{2}\ \underline{D}_l f(\xi).$$

This proves the theorem.

Theorem-2. (Rolle's theorem). Let $f(x)$ be

(*i*) Continuous in $[a,b]$

(*ii*) Schwarz differentiable in (a,b)

and (*iii*) $f(a) = f(b)$.

(*iv*) Let there exist a point x_0, $a < x_0 < b$ for which $f(x_0) = f(a)$.

(*v*) If $\overline{D}_r f(x) \geqslant \underline{D}_l f(x)$ for $x \in E = \{x;\ x \in [a,b], f(x) > f(a)\}$

there exists at least one point ξ in (a,b) such that

$$f^{(1)}(\xi) = 0.$$

Proof: Since $f(x)$ is continuous in $[a,b]$, it attains its upper bound at some point or points ξ in $[a,b]$. By hypothesis, E is non-void and $f(a) = f(b)$; so $a < \xi < b$.

Now $f^{(1)}(\xi)$ exists and so it follows from Theorem 1 that

$$\overline{D}_r f(\xi) \leqslant f^{(1)}(\xi) \leqslant \underline{D}_l f(\xi) \quad \ldots \quad (3)$$

Again, $f(\xi) > f(a)$; so, $\xi \in E$ and hence

$$\overline{D}_r f(\xi) \geqslant D_l f(\xi) \quad \ldots \quad (4)$$

Since $f(\xi)$ is the upper bound, we have $\bar{D}_r f(\xi) \leqslant O \leqslant \underline{D}_l f(\xi)$.

Utilising (3) and (4) we get $f^{(1)}(\xi)=O$. This proves the theorem.

Note 1. If there exist a point y_0, $a<y_0<b$, for which $f(y_0)<f(a)$ and if $\underline{D}_r f(x) \leqslant \bar{D}_l f(x)$ for $x \in F=\{x;\, x \in [a,b];\, f(x)<f(a)\}$ then it can be shown similarly with the assumptions (i), (ii) and (iii) that there exist at least one point η, $a<\eta<b$, such that $f^{(1)}(\eta)=O$.

Note 2. If both the sets E and F are void, then $f(x)$ is constant in $[a,b]$ and so $f^{(1)}(x)=f'(x)=O$ in (a,b).

Note 3. The assumption (v) is required only on a set of measure zero; because it is known that under the hypotheses of the theorem, $f(x)$ has a finite derivative almost everywhere [2].

Theorem 3. (Cauchy's mean value theorem). Let $f(x)$ and $g(x)$ be continuous in $[a,b]$ and Schwarz differentiable in (a,b). Let $g(a) \neq g(b)$ and $g^{(1)}(x) \neq O$ for $x \in (a,b)$. Let E and F denote the sets

$$E=\left\{x;\, x \in [a,b];\, f(x)-\frac{f(b)-f(a)}{g(b)-g(a)}\, g(x) > \frac{f(a)\,g(b)-f(b)\,g(a)}{g(b)-g(a)}\right\}$$

$$F=\left\{x;\, x \in [a,b];\, f(x)-\frac{f(b)-f(a)}{g(b)-g(a)}\, g(x) < \frac{f(a)\,g(b)-f(b)\,g(a)}{g(b)-g(a)}\right\}.$$

Suppose that

(i) $\bar{D}_r f(x) \geqslant \underline{D}_l f(x)$ for $x \in E$ & $\underline{D}_r f(x) \leqslant \bar{D}_l f(x)$ for $x \in F$.

and (ii) $\bar{D}_r g(x) \leqslant \underline{D}_l g(x)$ for $x \in E$ & $\underline{D}_r g(x) \geqslant \bar{D}_l g(x)$ for $x \in F$.

or (ii) $\underline{D}_r g(x) \geqslant \bar{D}_l g(x)$ for $x \in E$ & $\bar{D}_r g(x) \leqslant \underline{D}_l g(x)$ for $x \in E$.

according as $f(b)-f(a)$ and $g(b)-g(a)$ have the same or opposite signs.

There exists then at least one point ξ in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f^{(1)}(\xi)}{g^{(1)}(\xi)}.$$

Proof: If both the sets E and F are empty, then

$$f(x)-\frac{f(b)-f(a)}{g(b)-g(a)}\, g(x) = \frac{f(a)\,g(b)-f(b)\,g(a)}{g(b)-g(a)} \quad \text{for all } x \in [a,b]$$

$$\text{So, } f^{(1)}(x)-\frac{f(b)-f(a)}{g(b)-g(a)}\, g^{(1)}(x) = O \quad \text{for all } x \text{ in } (a,b).$$

The theorem is, therefore, proved in this case.

We suppose now that at least one of the sets E and F is not empty. For definiteness, suppose that E is not empty.

Let $$\varphi(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}\, g(x) \qquad \ldots (5)$$

Then $\varphi(x)$ is continuous in $[a,b]$, Schwarz differentiable in (a,b)

and $$\varphi(a) = \frac{f(a)\,g(b)-f(b)\,g(a)}{g(b)-g(a)} = \varphi(b) \qquad \ldots (6)$$

Since at each point x of E, $\varphi(x) > \varphi(a)$, utilising (6) it follows that E $\subset (a,b)$.

Denoting $\dfrac{f(b)-f(a)}{g(b)-g(a)}$ by K, we have from (5)

$$\varphi(x) = f(x) - K\, g(x).$$

$$\left.\begin{array}{lll} \text{Consequently,} & \overline{D}_r\,\varphi(x) \geqslant \overline{D}_r f(x) - K\, \overline{D}_r\, g(x) \text{ if } K > 0 \\ \quad\text{and} & \overline{D}_r\,\varphi(x) \geqslant \overline{D}_r f(x) - K\, \underline{D}_r\, g(x) \text{ if } K < 0 \\ \text{Similarly} & \underline{D}_l\,\varphi(x) \leqslant \underline{D}_l f(x) - K\, \underline{D}_l\, g(x) \text{ if } K > 0 \\ \quad\text{and} & \underline{D}_l\,\varphi(x) \leqslant \underline{D}_l f(x) - K\, \overline{D}_l\, g(x) \text{ if } K < 0 \end{array}\right\} \qquad (7)$$

From (7) it follows, by utilising (i) and (ii) that at each point x of E, $\overline{D}_r\,\phi(x) > \underline{D}_l\,\varphi(x)$. Hence by Theorem 2, there exist at least one point ξ, $a<\xi<b$, such that $\varphi^{(1)}(\xi) = 0$. *i.e.* $f^{(1)}(\xi) = \dfrac{f(b)-f(b)}{g(b)-g(a)}\, g^{(1)}(\xi)$

So, $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f^{(1)}(\xi)}{g^{(1)}(\xi)}$. This proves the theorem.

Note 4. It should be noted that in proving the above theorem, all the conditions in (i) and (ii), even if E and F are both non-empty, are not required. Because if E and F are non-empty, then we start with any one of them, say E and to achieve our result we need only those parts of the assumptions (i) and (ii) which are applicable to E.

Taking, in particular, $g(x)=x$ in the above theorem, we obtain the following.

Theorem 4. (Lagrange's mean value theorem). Let $f(x)$ be continuous in $[a,b]$ and Schwarz differentiable in (a,b). Let E and F denote the sets,

$$E = \left\{ x \; ; \; x \in [a,b] \; ; \; f(x) - \frac{f(b)-f(a)}{b-a} x > \frac{b f(a) - a f(b)}{b-a} \right\}.$$

$$F = \left\{ x \; ; \; x \in [a,b] \; ; \; f(x) - \frac{f(b)-f(a)}{b-a} < \frac{b f(a) - a f(b)}{b-a} \right\}$$

Suppose that $\overline{D}_r f(x) \geqslant \underline{D}_l f(x)$ for $x \in E$

and $\underline{D}_r f(x) \leqslant \overline{D}_l f(x)$ for $x \in F$

then there exist at least one point ξ in (a,b) such that

$$f(b) - f(a) = (b-a) f^{(1)}(\xi).$$

Theorem 5. Let $f(x)$ be continuous in $[a,b]$ and Schwarz differentiable in (a,b). Let E denote the set

$$E = \{x \; ; \; x \in [a,b] \; ; \; f(x) > \max [f(a), f(b)]\}.$$

If $\overline{D}r\, f(a) > O$ and $\underline{D}l f(b) < O$ and $\overline{D}r f(x) \geqslant \underline{D}l\, f(x)$ for $x \in E$ then there is a point ξ in (a,b) such that $f^{(1)}(\xi) = O$.

Proof: Since $\overline{D}_r f(a) > O$ and $\underline{D}_l f(b) < O$ there are points x, y, in (a,b) such that $f(x) > f(a)$ and $f(y) > f(b)$. Again as $f(x)$ is continuous in $[a,b]$, it assumes its upper bound at some point ξ, say. From the preceding, it follows that $a < \xi < b$ and also $\xi \in E$.

So, $$\overline{D}_r f(\xi) \geqslant \underline{D}_l f(\xi)$$

and by Theorem 1, since $f(\xi)$ is the upper bound of $f(x)$ in $[a,b]$,

$$\overline{D}_r f(\xi) \leqslant f^{(1)}(\xi) \leqslant \underline{D}_l f(\xi)$$

Since $f(\xi)$ is the upper bound, we have $\overline{D}_r f(\xi) \leqslant O \leqslant \underline{D}_l f(\xi)$

So, from above $f^{(1)}(\xi) = O$. This proves the theorem

Sec. B.

3. Let $f(x)$ be Schwarz differentiable at each point x of I and suppose that

$$\psi(x,h) = \frac{f(x+h) - f(x-h)}{2h} - f^{(1)}(x).$$

Then $f(x)$ is said to be *uniformly Schwarz differentiable* in $[a,b]$, if corresponding to the arbitrary $\epsilon > O$, there exists a $\delta > O$ (independent of x) such that $|\psi(x,h)| < \epsilon$, if $|h| < \delta$ for $x \in [a,b]$ and $x + h \in I$.

Since $f(x)$ is Schwarz differentiable in I, given $\epsilon > O$, there exists for each point $x \in [a,b]$, a $\delta(x) > O$, such that $|\psi(x,h)| < \epsilon$ for $|h| < \delta(x)$. It may happen that for a fixed value of $\epsilon > O$, $\delta(x)$ has no positive lower boundary in $[a,b]$. Then by a well-known theorem of Weierstrass [1] we can find at least one point η in $[a,b]$ and a neighbourhood of η on which $\delta(x)$ has no positive lower boundary.

We now introduce the following definition.

Definition. A point x at which for each $\epsilon > O$ there exists a neighbourhood in which $\delta(x)$ has a positive lower bound is said to be a point of uniform Schwarz differentiability of $f(x)$.

A point x in a neighbourhood of which $\delta(x)$ has no positive lower bound for some sufficiently small $\epsilon > O$ is said to be a point of nonuniform Schwarz differentiability of $f(x)$.

The above definitions are equivalent to the following :

A point η is a point of uniform or non-uniform Schwarz differentiability of $f(x)$ according as $\psi(x,h)$ has or has not the unique double limit zero as $x \to \eta$ and $h \to O$.

As an illustration, let $f(x) = x \sin \frac{1}{x}$, $x \neq O$

$= 0 \qquad , x = O$

This function is everywhere Schwarz differentiable, but $x = O$ is a point of non-uniform Schwarz differentiability of $f(x)$.

Let now ξ be an arbitrary point in $[a,b]$ and $(\xi - d, \xi + d)$, $d > O$, be a neighbourhood of ξ. Let p be any positive number. Let U denote the least upper bound of $\psi(x,h)$ for $x \in (\xi - d, \xi + d)$ and $|h| \leqslant p$. The function U is monotone non-increasing as $d \to O$ and $p \to O$. Consequently, it has a greatest lower bound as d and p tend to zero independently. We denote this greatest lower bound by $d(\xi)$.

The function $d(x)$ thus defined may have finite value or may be ∞ at some point or points. If $d(x) \neq O$ at some point ξ, then ξ is evidently a point of non-unifrrm Schwarz differentiability of $f(x)$ and at a point of uniform Schwarz differentiability, $d(x) = O$.

Theorem 6. $d(x)$ is upper semi continuous on $[a,b]$.

Proof: Let $\epsilon > O$ be arbitrary. We will show that if ξ is any point in $[a,b]$, there exists a neighbourhood of ξ at every point of which $d(x) < d(\xi) + \epsilon$. If possible, suppose the contrary. Then in every neighbourhood of ξ there are points η such that $d(\eta) \geqslant d(\xi) + \epsilon$.

In an arbitrarily small neighbourhood of such a point η there are points at which $|\psi(x,h)| > d(\xi) + \frac{\epsilon}{2}$ for sufficiently small values of $|h|$. This neighbourhood of η may be supposed to be contained in any preassigned neighbourhood of ξ. So, in every neighbourhood of ξ there are points at which $|\psi(x,h)| > d(\xi) + \frac{1}{2}\epsilon$ for sufficiently small values of $|h|$. But this contradicts the definition of $d(\xi)$. Hence the theorem follows.

Theorem 7. The set of points of $[a,b]$ at which $d(x)$ is infinite is closed.

Proof: Let $x_1, x_2, \ldots x_n, \ldots$ be an increasing sequence of positive numbers with $x_n \to +\infty$. Since by Theorem 6, $d(x)$ is upper semi-continuous, we have that the

set C_n of points where $d(x) \geqslant x_n$ is closed. Now, the set of points at which $d(x)$ is infinite is the inner limiting set of the sets C_n. This set is therefore closed.

Theorem 8. A necessary and sufficient condition that $d(x) \neq \infty$ at any point $x \in [a,b]$ is that there exists $h_1 \neq O$ such that $|\psi(x,h)|$ is bounded for all values of x in $[a,b]$ and for all h such that $|h| \leqslant |h_1|$.

Proof: The proof of sufficiency easily follows. Since if $|\psi(x,h)|$ is bounded so is $d(x)$.

To prove the necessity, choose $\epsilon > O$ arbitrary. From the definition of $d(x)$ it follows that for every point x' in $[a,b]$ there exists a neighbourhood D_x' of x' and a value $|h_x'| \neq O$ of $|h|$ such that

$$|\psi(x,h)| < d(x') + \epsilon.$$

for $x \in D_x'$ and for all h such that $|h| \leqslant |h_x'|$.

The set of neighbourhoods $\{D_x'\}$ is such that every point of the closed interval $[a,b]$ is interior to some neighbourhood D_x'.

Hence by Heine-Borel theorem, there exists a finite number of these neighbourhoods.

$$Dx_1', Dx_2', \ldots Dx'_m$$

which also have the same property. Let $|h_1'|$ be the least of $|hx'_1|$, $|hx'_2|$, ... $|hx'_m|$ and $d(x_1)$ be the greatest of $d(x_1')$, $d(x_2')$,.. $d(x_m')$. Then

$$|\psi(x,h)| < d(x_1) + \epsilon.$$

for all $x \in [a,b]$, provided $|h| \leqslant |h_1'|$. This proves the theorem.

Theorem 9. {Cf. Theorem 1 [3]}. If $f(x)$ is continuous and Schwarz differentiable in I and $d(\xi)=O$, $\xi \in [a,b]$, then $f^{(1)}(x)$ is continuous at ξ.

Proof: Since $d(\xi) = O$, for arbitrary $\epsilon > O$, there exists a neighbourhood D_1 of ξ such that at each point $x \in D_1$ we have $|\psi(x,h)| < \frac{1}{3}\epsilon$ for all sufficiently small values of $|h|$. We choose a particular value of h, say h_1 for which this is true. Then $|\psi(x,h_1)| < \frac{1}{3}\epsilon$ for all $x \in D_1$.

$$\text{Let } \Phi(x,h_1) = \frac{f(x+h_1)-f(x-h_1)}{2h_1}.$$

Then $\Phi(x,h_1)$ is continuous at ξ. So, there exists a neighbourhood D_2 of ξ such that

$$|\Phi(x,h_1)-\Phi(x,h_1)| < \tfrac{1}{3}\epsilon \text{ whenever } x_1, x_2 \in D_2.$$

Let $D = D_1 \cap D_2$. Then D is a neighbourhood of ξ. If α_1, α_2 be any two points of D, then

$$\begin{aligned} |f^{(1)}(\alpha_1) - f^{(1)}(\alpha_2)| &= |\Phi(\alpha_1,h_1) - \Phi(\alpha_2,h_1) - \psi(\alpha_1,h_1) + \psi(\alpha_2,h_1)| \\ &\leqslant |\Phi(\alpha_1,h_1) - \Phi(\alpha_2,h_1)| + |\psi(\alpha_1,h_1)| + |\psi(\alpha_2,h_1)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

So, $f^{(1)}(x)$ is continuous at ξ. This proves the theorem.

Corollary 1. If $f(x)$ is continuous and Schwarz differentiable in I and $\xi \in [a\ b]$ is a point of uniform Schwarz differentiability of $f(x)$ then $f^{(1)}(x)$ is continuous at ξ.

Corollary 2. If $f(x)$ is continuous and uniformly Schwarz differentiable in I then $f^{(1)}(x)$ is continuous in $[a,b]$.

Theorem 10. If $f(x)$ is continuous and uniformly Schwarz differentiable in I then $f(x)$ is uniformly differentiable in $[a,b]$.

Proof. For arbitrary $\epsilon < O$, there exists $\delta > O$ such that

$$\left| \frac{f(n+h)-f(x-h)}{2h} - f^{(1)}(x) \right| < \frac{\epsilon}{2} \qquad \dots (8)$$

whenever $x \in$ I and $|h| < \delta$.

Since by corollary 2 of Theorem 9, $f^{(1)}(x)$ is continuous and consequently uniformly continuous in $[a,b]$, there exists $\delta_1 > O$ such that

$$|f^{(1)}(x_1) - f^{(1)}(x_2)| < \frac{\epsilon}{2} \qquad \dots (9)$$

whenever $|x_2 - x_1| < \delta_1$ and $x_1, x_2 \in [a,b]$.

Choose an h such that $|h| < \delta' = \min(\delta, \delta_1)$. Then replacing x and h by $x + \frac{1}{2}h$ and $\frac{1}{2}h$ respectively in (8), we obtain

$$\left| \frac{f(x+h) - f(x)}{h} - f^{(1)}(x + \tfrac{1}{2}h) \right| < \tfrac{1}{2}\epsilon \qquad \dots (10)$$

So, $$\left| \frac{f(x+h)-f(x)}{k} - f^{(1)}(x) \right| = \left| \frac{f(x+h)-f(x)}{h} - f^{(1)}(x+\tfrac{1}{2}h) + f^{(1)}(x+\tfrac{1}{2}h) = f^{(1)}(x) \right|$$

$$< \frac{\epsilon}{2} + \left| f^{(1)}(x+\tfrac{1}{2}h) - f^{(1)}(x) \right| \text{ by (10)}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ by (9).}$$

Since $\epsilon < O$, is arbitrary, this shows that $f(x)$ is uniformly differentiable in $[a,b]$. This proves the theorem.

Utilising a result of Weinstock [6], we obtain the following corollary.

Corollary: If $f(x)$ is continuous and uniformly Schwarz differentiable in I, then $f^{(1)}(x)$ exists and coincides with $f^{(1)}(x)$ in $[a,b]$.

Moreover $f^{(1)}(x)$ is continuous in $[a,b]$.

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SOME IDENTITIES SATISFIED BY CARTAN'S CURVATURE TENSORS

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ABSTRACT

Cartan in one of his papers has defined three types of curvature tensors for a Finsler space and has deduced some identities involving these tensors. In this paper some more identities for the first and second of Cartan's curvature tensors have been deduced.

1. INTRODUCTION

Three different curvature tensors have been defined by E. Cartan | 1 |. Some identities are known for the curvature tensors. The object of this paper is to find some more identities satisfied by the first and the second of Cartan's curvature tensors. Similar identities for the third of Cartan's curvature tensors are known. First and second of Cartan's curvature tensors are given by

$$(1.1) \qquad S^i_{jkh} \overset{def}{=} 2\, A^i_{r[k} \quad A^r_{h]j} \;, \quad (1)$$

and

$$(1.2) \qquad P^i_{jkh} \overset{def}{=} F\, \dot{\partial}_h\, \Gamma^{*i}_{jk} + A^i_{jm} \quad A^m_{hk|r}\, l^r \; - A^i_{jh|k} \,,$$

or,

$$(1.2)\text{a} \quad P^i_{jkh} = A^i_{kh/j} - g^{im}\, A_{jkh|m} \; - A^i_{km} \quad A^m_{jh|r}\, l^r + A^m_{jk}\, A^i_{mh|r} l^r \;\; ;$$

where $\dot{\partial}_h$ and $k\,|$ denote $\partial/\partial \dot{x}^k$ and covariant derivative with respect to x^h .

These curvature tensors satisfy the following identities

$$(13) \qquad S^i_{j(hk)} = 0.$$

---

(1) Square and round brackets denote symmetric and skew symmetric parts, the indices enclosed in a rectangle are excluded from a symmetric and alternating part, e.g. $T_{[\,i\,\boxed{j}\,k\,]} = T_{ijk} - T_{kji}$ and $T_{(i\,\boxed{j}\,k)} = T_{ijk} + T_{kji}$

If we write

(1.4) $$S_{ijkh} \overset{def}{=} g_{rj} S^{r}_{ikh} = 2\, A_{mj[k} A^{m}_{h]i}$$

and

(1.5) $$P_{ijkh} \overset{def}{=} g_{rj} P^{r}_{ikh}$$

then we put

(1.6) $$P_{ijkh} = A_{jkh/i} - A_{ikh/j} - A_{jkm} A^{m}_{ih/r}\, l^{r} + A^{m}_{ik} A_{jmh/r}\, l^{r} .$$

We also get

(1·7) $$S_{(ij)\,hk} = 0,$$

and

(1.8) $$P_{(ij)hk} = 0,$$

together with

(1.9) $$P_{ijkh}\, (x,\dot{x})\, \dot{x}^{i} = A_{jkh/i}\, (x,\dot{x})\, \dot{x}^{i} .$$

From the definition, it is obvious that

(1.10) $$P^{i}_{jkh}\, \dot{x}^{h} = 0,$$

and

(1.11) $$S^{i}_{jkh}\, \dot{x}^{h} = 0.$$

2. IDENTITIES FOR THE FIRST OF CARTAN'S CURVATURE TENSOR

In addition to the above identities these tensors satisfy the following identities also :

(2.1) $$S^{i}_{[j\,kh]} = 0,$$

Multiplying (2.1) by $g_{ir}\,(x, \dot{x})$, we get

(2·2) $$S_{i\,j\,[\,\underline{r}\,\,kh\,]} = 0.$$

or,

(2.3) $$S_{i\,[jkh]} = 0$$

by virtue of (1.7).

From (2·3) and (1.3), we get

(2·4) $$2\, S_{i\,[\,j\,|\overline{k}|\,h\,]} = S_{ikjh}$$

and similarly, we get

(2·5) $$2\, S_{k\,[\,h\,|\overline{i}|\,j\,]} = S_{kihj}$$

Adding (2.4) and (2.5), we get

(2·6) $$S_{i\,[\,j\,|\overline{k}|\,h]} + S_{k\,[h\,|\overline{i}|\,j]} = S_{ikjh}$$

By the definition it is obvious that

(2.7) $$S_{ijkh} = S_{khij}\,.$$

If we contract the equation (1.1) with respect to i,h, we get

(2·8) $$S_{jk} \overset{def}{=} S_{jki}^{\;\;\;i} = A^{i}_{kr}\, A^{r}_{ji} - A^{i}_{ir}\, A^{r}_{jk}\,.$$

By interchanging j,k in the above equation and subtracting it from the above equation, we get

(2·9) $$S_{[jk]} = 0.$$

If we contract the equation (1·1) with respect to $i\,j$, we have

(2·10) $$S^{i}_{ikh} = 0.$$

3 IDENTITIES FOR THE SECOND OF CARTAN'S CURVATURE TENSORS

From the definition of P^{i}_{jkh}, we can prove that

(3.1) $$P^{i}_{[jkh]} = 0.$$

Multiplying (3.1) by g_{ir}, we get

(3.2) $$P_{[j\,|\overline{r}|\,kh]} = 0, \text{ i.e. } P_{r[\,jkh]} = 0.$$

From the definition (1.2)a, we get

$$(3.3) \qquad P^{i}_{j[kh]} = (A^{i}_{m[k}\ A^{m}_{k]}\ )_{/r}\ l^{r}$$

By virtue of (1.1), we obtain

$$(3.4) \qquad 2\ P^{i}_{j[kh]} = S^{i}_{jhk/r}\ l^{r}\ .$$

Multiplying (3.4) by g_{ip}, we find that

$$(3.5) \qquad 2\ P_{jp[kh]} = S_{jphk/r}\ l^{r}\ , \text{ as } g_{ip/r} = 0.$$

By subtracting the values of P_{ijkh} and P_{ihkj}, we find that

$$(3\cdot6) \qquad 2\ P_{i\,[j\,|\underline{\overline{k}}|\,h]} = P_{hjki}\,.$$

Similarly, we have

$$(3\cdot7) \qquad 2\ P_{k\,[j\,|\underline{\overline{i}}|\,h]} = P_{hjik}\,.$$

Adding (3.6) and (3.7), we get

$$(3.8) \qquad P_{i\,[j\,|\underline{\overline{k}}|\,h]} + P_{k\,[j\,|\underline{\overline{i}}|\,h]} = P_{hj(ki)}$$

If we interchange the pair of indices i,j and k,h, we obtain

$$(3.9) \qquad P_{ijkh} = P_{khij} + M_{hijk}$$

where

$$(3.10) \quad M_{hijk} = A_{hjk/i} + A_{ijk/h} + A_{him}\ A^{m}_{kj/p}\ l^{p} -$$

$$- (A_{jhi/k} + A_{khi/j} + A_{jkm}\ A^{m}_{hi/p}\ l^{p}\ ).$$

From (3.10), it is evident that

$$M_{hijk} = M_{ihjk} = M_{hikj}$$

In particular, if we multiply (3.9) by $\dot{x}^{i}$, we get

$$(3.11) \qquad (P_{ijkh} - P_{khij})\ \dot{x}^{i} = A_{hjk/i}\ \dot{x}^{i}$$

Contracting the equation (1.2)a with respect to i,j, we get

$$P^{i}_{ikh} = 0. \tag{3.12}$$

By contracting (1.2)a with respect to i,h, we get

$$P_{jk} = P^{i}_{jki} = A^{i}_{ki/j} - g^{im} A_{jki/m} - A^{i}_{km} A^{m}_{ji/r} l^{r} + A^{m}_{jk} A^{i}_{mi/r} l^{r} . \tag{3.13}$$

Interchanging j,k in (3.13) and then by subtracting it from (3.13), we get

$$P_{[jk]} = A^{i}_{i[k/j]} + A^{i}_{m[j} A^{m}_{k]i/r} l^{r} . \tag{3.14}$$

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DIFFERENTIAL EQUATIONS OF LAURICELLA'S F_D*

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ABSTRACT

In the course of a study of the 'four-term' differential equation

$$\sum_{\lambda=0}^{3} (-x)^{\lambda} f_{\lambda}(\delta)\, z = 0 \qquad \left(\delta = x \frac{d}{dx} \right)$$

the author in collaboration with S. Saran established the theorem that a function z which satisfies the system of partial differential equations

$$\theta_i f(\delta) z = p_i x (\theta_i + \alpha_i)\, g(\delta) z \qquad (i = 1,2,3)$$

where $\theta_i = p_i \dfrac{\partial}{\partial p_i}$ and $\sum_{i=1}^{3} \theta_i = \delta$, is a solution of the ordinary equation

$$\delta f(\delta) f(\delta-1) f(\delta-2) z$$
$$- x\Big[p_1(\delta+\alpha_1)+p_2(\delta+\alpha_2)+p_3(\delta+\alpha_3)\Big] f(\delta) f(\delta-1)\, g(\delta) z$$
$$+ x^2\Big[p_1 p_2(\delta+\alpha_1+\alpha_2)+p_2 p_3(\delta+\alpha_2+\alpha_3)+p_3 p_1(\delta+\alpha_3+\alpha_1)\Big] f(\delta)\, g(\delta)\, g(\delta+1) z$$
$$- p_1 p_2 p_3\, x^3 (\delta+\alpha_1+\alpha_2+\alpha_3)\, g(\delta)\, g(\delta+1)\, g(\delta+2)\, z = 0.$$

Also the linear ordinary differential equation associated with Lauricella's triple series

$$F_D(\alpha, \beta_1, \beta_2, \beta_3; \gamma; px, qx, rx)$$

was shown to possess one hundred and twenty solutions in terms of F_D, there being no logarithmic solution at any of the five singularities.

In the present paper the author discusses in details two interesting exceptions to the general statement that in the neighbourhood of a regular singularity at the origin the equation can be solved in terms of the functions F_D themselves.

*Read at the thirty-fourth annual session of the Academy.

§ 1

In an earlier paper [4] Srivastava and Saran studied the 'four-term' differential equation

(1.1) $$\left[f(\delta)-x\,g(\delta)+x^2\,h(\delta)-x^3\,k(\delta)\right]z=0 \qquad \left(\delta=x\frac{d}{dx}\right)$$

with special reference to the linear ordinary differential equations associated with certain triple hypergeometric functions

$$F(px,\,qx,\,rx),$$

where p, q, r are parameters and x is the variable of differentiation.

We proved the following theorem.

Theorem. 'A function z which satisfies the system of partial differential equations

(1·2) $$\begin{cases} \theta\, f(\theta+\phi+\psi)z=px\,(\theta+a)\,g(\theta+\phi+\psi)z \\ \phi\, f(\theta+\phi+\psi)z=qx\,(\phi+b)\,g(\theta+\phi+\psi)z \\ \psi\, f(\theta+\phi+\psi)z=rx\,(\psi+c)\,g(\theta+\phi+\psi)z \end{cases}$$

where $\theta,\ \phi\ \psi=p\dfrac{\partial}{\partial p},\ q\dfrac{\partial}{\partial q},\ r\dfrac{\partial}{\partial r}$, and $\theta+\phi+\psi=\delta$,

also satisfies the ordinary equation

(1.3) $$\begin{aligned} &\delta\, f(\delta)\, f(\delta-1)\, f(\delta-2)z \\ &-x\left[p(\delta+a)+q(\delta+b)+r(\delta+c)\right]f(\delta)\, f(\delta-1)\, g(\delta)z \\ &+x^2\left[pq(\delta+a+b)\,(\delta+a+b)+qr(\delta+b+c)+rp(\delta+c+a)\right]f(\delta)\, g(\delta)\, g(\delta-1)z \\ &-pqr\, x^3\,(\delta+a+b+c)\, g(\delta)\, g(\delta+1)\, g(\delta+2)z=0, \end{aligned}$$

there being nothing in the argument that precludes the existence of solutions of (1.3) which are not solutions of (1.2).

In particular, the Lauricella's triple hypergeometric function [3]

$$F_D(a,\,b_1,\,b_2,\,b_3;\,c;\,px,\,qx,\,rx)$$

$$=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{l+m+n}\,(b_1)_l\,(b_2)_m\,(b_3)_n}{(c)_{l+m+n}}\cdot\frac{(px)^l}{l!}\cdot\frac{(qx)^m}{m!}\cdot\frac{(rx)^n}{n!}$$

is a solution of

$$
\begin{aligned}
&\delta(\delta+c-1)(\delta+c-2)(\delta+c-3)z\\
(1.4)\qquad &-x\Big[p(\delta+b_1)+q(\delta+b_2)+r(\delta+b_3)\Big](\delta+c-1)(\delta+c-2)(\delta+a)z\\
&+x^2\Big[pq(\delta+b_{12})+qr(\delta+b_{23})+rp(\delta+b_{31})\Big](\delta+c-1)(\delta+a)(\delta+a+1)z\\
&-pqrx^3(\delta+b_{123})(\delta+a)(\delta+a+1)(\delta+a+2)z=0,
\end{aligned}
$$

where, for the sake of brevity,

$$b_{123}=b_1+b_2+b_3,\; b_{23}=b_2+b_3, \text{ etc.}$$

Put

$$z=x^{1-c}\,z',$$

and suppress the accent, and (1·4) transforms into

$$
\begin{aligned}
&\delta(\delta-1)(\delta-2)(\delta+1-c)z\\
&-x\Big[p(\delta+b_1+1-c)+q(\delta+b_2+1-c)+r(\delta+b_3+1-c)\Big]\delta(\delta-1)(\delta+a+1-c)z\\
(1.5)\qquad &+x^2\Big[pq(\delta+b_{12}+1-c)+qr(\delta+b_{23}+1-c)+rp(\delta+b_{31}+1-c)\Big]\delta(\delta+a+1-c)\\
&\qquad\cdot(\delta+a+2-c)z\\
&-pqr\,x^3(\delta+b_{123}+1-c)(\delta+a+1-c)(\delta+a+2-c)z=0.
\end{aligned}
$$

The singularities of (1.5), viz.

$$x=0,\ p^{-1},\ q^{-1},\ r^{-1},\ \infty,$$

have the same character [2], and we observe that the integral differences of the exponents 0, 1, 2 at the origin do not give rise to a logarithmic solution. The equation (1.5), and consequently (1.4), will therefore have no logarithmic solutions.

§ 2.

Out of the one hundred and twenty solutions of (1.4), detailed in [4], we may select as solutions valid in the neighbourhood of the origin the following functions :

$$z_1 = F_D(a, b_1, b_2, b_3; c; px, qx, rx)$$

$$(2.1)\qquad = \sum_{n=0}^{\infty}\frac{(a)_n\,(b_3)_n}{(c)_n}\cdot\frac{(rx)^n}{n!}\;F\Big(-n, b_1, b_2; 1-n-b_3; \frac{p}{r}, \frac{q}{r}\Big)$$

$$z_{10} = x^{1-c}\, F_D\left(b_{123}+1-c_1\, a+1-c_1\, b_2\, b_3;\, 2+b_{23}-c;\, px, \frac{p}{q}, \frac{p}{r}\right)$$

$$(2.2) \qquad = x^{1-c} \sum_{n=0}^{\infty} \frac{(b_{123}+1-c)_n\, (a+1-c)_n}{(2+b_{22}-c)_n} \cdot \frac{(px)^n}{n!}$$

$$\cdot\, F_1\left(b_{123}+n+1-c,\, b_3,\, b_3,\, 2+b_{23}+n-c;\, \frac{p}{q}, \frac{p}{r}\right)$$

$$z_{30} = x^{1-c}\quad F_D\left(b_{123}+1-c,\, b_1,\, a+1-c,\, b_3;\, 2+b_{31}-c;\, \frac{q}{p}, qx, \frac{q}{r}\right)$$

$$(2.3) \qquad = x^{1-c} \sum_{m,n=0}^{\infty} \frac{(b_{123}+1-c)_{m+n}\quad (a+1-c)\, (b_3)_n}{(2+b_{31}-c)_{m+n}} \cdot \frac{(qx)^m}{m!} \cdot \frac{(q/r)^n}{n!}$$

$$\cdot\ {}_2F_1\left(b_{123}+m+n+1-c,\, b_1;\, 2+b_{31}+m+n-c;\, \frac{q}{p}\right)$$

$$z_{21} = x^{1-c}\, F_D\left(b_{123}+1-c,\, b_1,\, b_2,\, a+1-c;\, 2+b_{12}-c;\, \frac{r}{p}, \frac{r}{q}, rx\right)$$

$$(2.4) \qquad = x^{1-c} \sum_{m,n=0}^{\infty} \frac{(b_{123}+1-c)_{m+n}\, (a+1-c)_m\quad (b_1)_n}{(2+b_{12}-c)_{m+n}} \cdot \frac{(rx)^m}{m!} \cdot \frac{(r/p)^n}{n!}$$

$$\cdot\ {}_2F_1\left(b_{123}+m+n+1-c,\, b_2;\, 2+b_{12}+m+n-c;\, \frac{r}{q}\right).$$

By an appeal to the theory of analytic continuation (see [1], *p.* 26) from (2.3) we obtain a further solution

$$(2.5) \qquad z'_{20} = x^{1-c} \sum_{n=0}^{\infty} \frac{(a+1-c)_n\quad (b_{23}+1-c)_n}{(2+b_{,}-c)_n} \cdot \frac{(qx)^n}{n!}$$

$$\cdot\ {}_2F_1\left(=n,\, b_{,};\, c-a-n;\, \frac{1}{rx}\right)\, {}_2F_1\left(c-b_2-n-1,\, b_1;\; c-b_{23}-n;\, \frac{p}{q}\right),$$

and from (2.4) we similarly have

$$(2.6) \qquad z'_{21} = x^{1-c} \sum_{n=0}^{\infty} \frac{(a+1-c)_n\quad (b_{21}+1-c)_n}{(2+b_1-c)_n} \cdot \frac{(rx)^n}{n!}$$

$$\cdot\ {}_2F_1\left(-n,\ b_1;\ c-a-n;\ \frac{1}{px}\right)\ {}_2F_1\left(c-b_1-n-1,\ b_2;\ c-b_{31}-n;\ \frac{q}{r}\right).$$

When $|p| < |q| < |r|$, it is natural to adopt $z_1, z_{19}, z'_{0}, z'_{21}$ as the appropriate solutions. On the other hand if $|p| > |q| > |r|$, the appropriate solutions may be derived from these by suitable permutations of p, q, r and b_1, b_2, b_3.

§ 3.

We have seen that (1.4) may in general be solved in terms of Lauricella's functions F_D, but to this general statement we note here two exceptions of some interest.

When $p=q=r$, which is a singularity of the system

$$(3.1)\qquad \begin{cases} \theta(\theta+\phi+\psi+c-1)z = px\,(\theta+\phi+\psi+a)\,(\theta+b_1)z \\ \phi(\theta+\phi+\psi+c-1)z = qx\,(\theta+\phi+\psi+a)\,(\phi+b_2)z \\ \psi(\theta+\phi+\psi+c-1)z = rx\,(\theta+\phi+\psi+a)\,(\psi+b_3)z \end{cases}$$

associated with F_D, we may without loss of generality assume

$$p=q=r=1.$$

The point $x=1$ is then a 'confluent' singularity of (1.4) and (1.5), and that it is regular can be readily seen. Therefore it remains true that solutions valid in its neighbourhood are free of logarithms. An inspection of the solutions obtained in the preceding section reveals the fact that z_{19}, z_{20} and z_{21} are save, for a constant factor, identical. We may now write the equation (1.4) in the form

$$\Big[(\delta+c-3)-x(\delta+a)\Big]\Big[(\delta+c-2)-x(\delta+a)\Big]$$

$$\cdot\Big[\delta(\delta+c-1)-x(\delta+a)(\delta+b_1+b_2+b_3)\Big]z=0$$

whose obvious solutions

$$z_A={}_2F_1(a,\ b_1+b_2+b_3;\ c;\ x)$$

$$z_B=x^{1-c}\ {}_2F_1(b_1+b_2+b_3+1-c,\ a+1-c;\ 2-c;\ x)$$

are the degenerate forms of z_1 and z_{19}.

Since the equation

$$(\delta+c-\lambda)\,y=x\,(\delta+a)\,y$$

is satisfied by

$$y = x^{\lambda - c} \quad (1-x)^{c-a-\lambda} ,$$

therefore the two particular integrals of

$$\delta(\delta + c - 1)\, z - x\, (\delta + a)\ (\delta + b_1 + b_2 + b_3)\ z = C\, x^{\lambda - c} \quad (1 - x)^{c - a - \lambda}$$

for $\lambda = 2$ and $\lambda = 3$, where C is any convenient constant, provide the third and the fourth solutions of (1.4), *viz.*

$$z = \int^{x} (1-\xi)^{b_{123}-\lambda} \left[z_A(\xi)\, z_B(x) - z_A(x)\ \ z_B(\xi) \right] \,d\xi.$$

where $\lambda = 2,3.$

In another case of failure when $c = a + 1$, the solutions z_{19}, z'_{20}, z'_{21} all reduce to x^{1-c} and we thus require an additional pair of solutions of (1.4). We consider the equivalent equation (1.5) which may be re-written

$$\begin{aligned}(\delta-1)\,(\delta-2)\ \Big\{ \delta + 1 - c - x \Big[p\,(\delta + b_1 + 1 - c) + q\,(\delta + b_2 + 1 - c) + r(\delta + b_3 + 1 - c) \Big] \\ + x^2 \Big[pq\,(\delta + b_{12} + 1 - c) + qr\,(\delta + b_{23} + 1 - c) + rp\,(\delta + b_{31} + 1 - c) \Big] \\ - \ pqrx^3 \ \ (\delta + b_1 + b_2 + b_3 + 1 - c) \Big\} \quad \delta z = 0.\end{aligned}$$

An obvious solution of this equation is

$$z = \text{const.},$$

which leads to x^{1-c} as a solution of (1.4). To obtain others we write

$$\delta z = x^{c-1}\ W,$$

and we have

$$\begin{aligned}(\delta + c - 2)\ \ (\delta + c - 3)\ \ \Big\{ \delta - x \Big[p\,(\delta + b_1) + q\,(\delta + b_2) + r\,(\delta + b_3) \Big] \\ + x^2 \Big[pq\,(\delta + b_{12}) + qr\,(\delta + b_{23}) + rp\,(\delta + b_{31}) \Big] \\ - \ pqr\ x^3\,(\delta + b_1 + b_2 + b_3) \Big\} W = 0,\end{aligned}$$

The equation within the crooked bracket yields a solution of (3.3) in the form*

$$W = (1-px)^{-b_1} \ (1-qx)^{-b_2} \ (1-rx)^{-b_3} \ ,$$

and this leads to

$$z = \int^x x^{c-2} \ (1-px)^{-b_1} \ (1-qx)^{-b_2} \ (1-rx)^{-q_3} \ dx$$

$$= \frac{x^{c-1}}{(c-1)} \ F_D \ (c-1, b_1, b_2, b_3; c; px, qx, rx) \ ,$$

which is an appropriately modified z_1 as a solution of (1.4).

Now the equations

$$(\delta+c-2) \, u = O = (\delta+c-3) \, v$$

are satisfied by

$$u = x^{2-c} \quad \text{and} \quad v = x^{3-c}$$

respectively. Therefore the second and the third solutions of (3.3) are

$$W = (1-px)^{-b_1} \ (1-qx)^{-b_2} \ (1-rx)^{-b_3}$$

$$\cdot \int x^{\mu+1-c} \ (1-px)^{b_1-1} \ (1-qx)^{b_2-1} \ (1-rx)^{b_3-1} \ dx$$

where $\mu = O$, 1, and these lead us to two series solutions of (1.4), *viz.*

$$z = \int^x x^{c-2} \ (1-px)^{-b_1} \ (1-qx)^{-b_2} \ (1-rx)^{-b_3} \ .$$

$$\cdot \left\{ \int x^{\mu=1+c} \ (1-px)^{b_1-1} \ (1-qx)^{b_2-1} \ (1-rx)^{b_3-1} \ dx \right\} dx$$

for $\mu = O$ and $\mu = 1$.

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*Cf. eqn. (2.2) in [4].

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ON GENERALIZED LAPLACE TRANSFORM

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ABSTRACT

In this paper, some theorems on Varma and Whittaker transforms defined as

$$\phi(p) = p\int_0^\infty (pt)^{m-1/2} e^{pt/2} W_{k,m}(pt)\, h(t)\, dt$$

and

$$\phi(p) = p\int_0^\infty (2pt)^{-1/4} W_{k,m}(2pt)\, h(t)\, dt$$

respectively, are obtained. Particular cases of the theorems are the results due to Rathie and Bose. As applications of the theorems, some infinie integrals involving Whittaker function and Gouss's hypergeometric function are evaluated.

1. INTRODUCTION

The classical Laplace transform

$$\phi(p) = p\int_0^\infty e^{-pt} h(t)\, dt \qquad \dots (1.1)$$

has been generalized by VARMA (9) in the forms

$$f(p) = p\int_0^\infty (pt)^{m-1/2} W_{k,m}(pt) e^{-pt/2} h(t) dt \qquad \dots (1.2)$$

and

$$\psi(p) = p\int_0^\infty (2pt)^{-\frac{1}{4}} W_{\lambda,\mu}(2pt)\, h(t)\, dt \qquad \dots (1.3)$$

as Varma transform of Second kind and Varma transform of first kind respectively. (1.3) is also called as Whittaker transform.

Relation (1.2) and (1.3) can be reduced to (1.1) by putting $k+m = \frac{1}{2}$ and $\lambda = \pm\mu = \frac{1}{4}$ respectively by virtue of the identities :

$$W_{\frac{1}{2}-m,\, \pm m}(x) \equiv x^{\frac{1}{2}-m} e^{-x/2}$$

and

$$W_{\frac{1}{4},\, \pm\frac{1}{4}}(x) \equiv x^{\frac{1}{4}} e^{-x/2}$$

Throughout this paper we shall denote the relation (1.2) as

$$f(p) \overset{V_2}{\underset{k,m}{=}} h(t),$$

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the relation (1.3) as

$$\psi(p) \overset{V_1}{\underset{\lambda,\mu}{=}} h(t)$$

and the relation (1.1) as usual shall be denoted as

$$\phi(p) \doteqdot h(t)$$

2. Theorem 1. If

$$\phi(p) \overset{V_2}{\underset{k,m}{=}} \psi(t)$$

and

$$p^{\nu+1}\,\psi(1/p) \overset{V_1}{\underset{\lambda,\mu}{=}} g(t)$$

then

$$\phi(p) = 2^{\nu}\,(2p)^{m+\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(\frac{1}{2})^r}{\lfloor r} \int_0^{\infty} x^{m+\nu+\frac{1}{2}}\, g(x)$$

$$G^{4,0}_{2,4}\left(2px \,\middle|\, \begin{matrix} 1-k, \frac{1}{4}-\lambda-m-\nu+r \\ \frac{1}{2}\pm m, -\frac{1}{4}\pm\mu-m-\nu+r \end{matrix}\right) dx\,, \qquad \ldots (2.1)$$

provided $R\,(m+\nu+\lambda\pm m+5/4)>0$, $R(5/4+k+m\pm\mu+\nu)>0$, $R(p)>0$, $R(\rho+m+\nu+3/2)<0$, where $g(x) = 0\,(x^{\rho})$ for large x and $R\,(\sigma+m+\nu+\beta+3/2)>0$ where $g(x) = (x^{\sigma})$ for small x and $\beta = \min.\,(\frac{1}{2}\pm m, -\frac{1}{4}\pm\mu-m-\nu)$.

PROOF : By hypothesis

$$\phi(p) = p\int_0^{\infty} (pt)^{m-\frac{1}{2}}\, e^{-pt/2}\, W_{k,m}(pt)\,\psi(t)\,dt \qquad \ldots (2.2)$$

and

$$\psi\left(\frac{1}{p}\right) = p^{-\nu} \int_0^{\infty} (2pt)^{-\frac{1}{4}}\, W_{\lambda,\mu}(2pt)\, g(t)\,dt. \qquad \ldots (2.3)$$

Substituting the value of $\psi(t)$ from (2.3) in (2.2) and on changing the order of integration we get

$$\phi(p) = 2^{-\frac{1}{4}} p^{m+\frac{1}{2}} \int_0^{\infty} x^{-\frac{1}{4}}\, g(x)\,dx \int_0^{\infty} t^{m+\nu-\frac{1}{4}}\, e^{-pt/2}\, W_{k,m}(pt)\, W_{\lambda,\mu}\left(\frac{2x}{t}\right) dt \qquad \ldots (A)$$

$$=2^{-\frac{1}{2}} p^{m+\frac{1}{2}} \sum_{r=0}^{\infty} \frac{1}{r} \int_0^{\infty} x^{r-\frac{1}{2}} g(x)\, dx \int_0^{\infty} t^{m+\nu-r-\frac{1}{2}} e^{-\left(\frac{pt}{2}+\frac{x}{t}\right)}$$

$$W_{k,m}(pt)\, W_{\lambda,\mu}\left(\frac{2x}{t}\right) dt \qquad \ldots\ldots \text{(B)}$$

on expanding $e^{x/t}$ and changing the order of integration and summation. Now integrating term by term with the help of the result (6., p. 412)

$$\int_0^{\infty} t^{\rho-1} e^{-\frac{1}{2}\left(\frac{x}{\alpha}+\frac{\beta}{x}\right)} W_{k,\mu}\left(\frac{x}{\alpha}\right) W_{\lambda,\nu}(\beta/x)\, dx = \beta^{\rho}$$

$$G_{24}^{40}\left(\frac{\beta}{\alpha}\ \middle|\ \begin{matrix} 1-k,\ 1-\lambda-\rho \\ \frac{1}{2}\pm\mu,\ \frac{1}{2}\pm\nu-\rho \end{matrix}\right)$$

where R $(a) > 0$ and R $(\beta) > 0$, we obtain the theorem.

The above proof involves two inversions which have to the justified, viz. in steps (A) and (B).

The first inversion is justified, under the conditions stated, by virtue of Dela vallee Poussin's theorem (3., p. 504), since both x- and t- integrals converge absolutely and the double integral exists as the asymptotic behaviour of Whittaker function is as follows :

$$\left.\begin{matrix} \text{As } x \to \infty,\ W_{k,m}(x) = 0\,(x^{k} e^{-x/2}) \\ \text{and} \\ \text{as } x \to 0,\ W_{k,m}(x) = 0\,(a x^{m+\frac{1}{2}} + b x^{-m+\frac{1}{2}}) \end{matrix}\right\} \qquad \ldots (2.4)$$

The inversion in the step (B) is valid since each term of the series

$$\frac{(xt)^r}{Lr}$$

is continuous and

$$\sum_{r=0}^{\infty} \left| \frac{(xt)^r}{Lr} \right|$$

converges unifromly in the arbitrarily large finite interval $0 < t < \alpha$;

also $t^{m+\nu-\frac{1}{2}} e^{pt/2 - x/t} W_{k,m}(pt)\, W_{\lambda,\mu}(2x/t)$ is bounded and integrable in the interval $0 < t < \alpha$.

COROLLARY 1.

If the above theorem if we take $\lambda = \pm\mu = \frac{1}{4}$, $\nu = k-m-3/2$ and use the relation

$$G_{02}^{20}\left(x \mid a,b\right) = 2\, x^{(a+b)/2}\, K_{a-b}(2\sqrt{x})$$

we obtain the theorem given by RATHIE [7., p. 67]

COROLLARY 2.

Taking $k+m=\frac{1}{2}$, we get the theorem as

If

$$\phi(p) \doteqdot \psi(t)$$

and

$$p^{\nu+1}\,\psi(1/p) \overset{V_1}{\underset{\lambda,\mu}{=}} g(t)$$

then

$$\phi(p) = 2^{\nu}\,(2p)^{\frac{3}{4}} \sum_{r=0}^{\infty} \frac{p^r}{Lr} \int_0^{\infty} x^{\nu+r+\frac{3}{4}}\, G_{13}^{30}\left(2px \,\middle|\, \begin{matrix} -\lambda-\nu \\ \frac{1}{4}-r,\ \pm\mu-\frac{1}{2}-\nu \end{matrix}\right) g(x)\,dx \quad (2{\cdot}5)$$

provided $R(\lambda+\nu+3/2)>0$, $R(\nu\pm\mu+7/4)<0$, $R(p)>0$, $R(\rho+\nu+7/4)<0$,

where $g(x) = 0\,(x^{\rho})$ for large x and $R(\sigma+\nu+\beta+7/4)>0$ where $g(x)=0\,(x^{\sigma})$ for small x and $\beta = \min\left(\frac{1}{4}, -\frac{1}{2}\pm\mu-\nu\right)$.

EXAMPLE :

If we take (1., p. 13)

$$g(x) = t^{\sigma}\, e^{-qt},$$

then $$p^{\nu+1}\,\psi(1/p) \overset{V_1}{\underset{\lambda,\mu}{=}} g(t) = \frac{\Gamma(\sigma\pm\mu+5/4)}{2\,(2p)^{\sigma}\,\Gamma(\sigma-k+7/4)}\ {}_2F_1\left[\begin{matrix}\sigma\pm\mu+5/4\\ \sigma-k+7/4\end{matrix};\ \frac{1}{2}-\frac{q}{2p}\right]$$

where $R(\sigma\pm\mu+5/4)>0$, $R(p) > R(p_o)>0$ and $|p| > |q|$. ... (2.6)

Using the value of $g(t)$ in (2.1) and solving the integral, we get

$$\phi(p) = \frac{2^{\nu}\,(2p)^{m+\frac{1}{2}}}{q^{\sigma+m+\nu+3/2}} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)^r}{Lr}\, G_{3,4}^{4,1}\left(\frac{2p}{q} \,\middle|\, \begin{matrix} -\sigma-m-\nu-\frac{1}{2},\ 1-k,\ \frac{1}{2}-\lambda-m-\nu+r \\ \frac{1}{2}\pm m,\ -\frac{1}{4}\pm\mu-m-\nu+r \end{matrix}\right)$$

where $R(p > R(p_o) > 0$ and $|p| > |q|$. ... (2·7)

Now using the value of $\psi(p)$ form (2.6) in the relation (2.2) and equating this integral with R. H. S. of (2.7) and on adjusting the parameters, we get.

$$\int_{0}^{\infty} t^{\nu-1} e^{-pt/2} W_{k,m}(pt)\; {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} ; \tfrac{1}{2} - \frac{qt}{2}\right] dt$$

$$= \frac{2^{\nu}\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,q^{\nu}} \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r}{r!}\, G^{4,1}_{3,4}\left(\frac{2p}{q}\,\middle|\,\begin{matrix} 1-\nu,\ 1-k,\ c-\nu+r \\ \tfrac{1}{2}\pm m,\ a-\nu+r,\ b-\nu+r \end{matrix}\right) \qquad \ldots (208)$$

where $R(a)>0$, $R(b)>0$, $R(p) > R(p_0)>0$ and $|p| > |q|$.

In this relation if we take $b=c$ and use the result (4., p. 213)

$$G^{m,n}_{p,q}\left(\lambda x\,\middle|\,\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = \lambda^{b_1} \sum_{r=0}^{\infty} \frac{1}{r!}(1-\lambda)^r\, G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix} a_1, \ldots, a_p \\ b_1+r, b_2, \ldots, b_q \end{matrix}\right) \qquad \ldots (2{\cdot}9)$$

we get the known result (6., p. 237).

3. **THEOREM 2.** If

$$\phi(p) \overset{V_2}{\underset{k,m}{=}} \psi(t^n)$$

and

$$p^{1-\nu/n}\,\psi(p) \overset{V_1}{\underset{\lambda,\mu}{=}} g(t)$$

then

$$\phi(p) = \frac{(2\pi)^{\frac{1-n}{2}}\, p^{\frac{n}{4}-\nu}}{2^{\frac{1}{4}}\, n^{n/4-\nu-k-m}} \sum_{r=0} \frac{(\frac{1}{2})_r}{r!} \int_0^{\infty} x^{-\frac{1}{4}} g(x)\, G^{2,2n}_{2n+1,\,n+2}$$

$$\left(\frac{2xn^n}{p^n}\,\middle|\,\begin{matrix} \Delta(n, n/4-\nu),\ \Delta(n, n/4-\nu-2m),\ 1-\lambda+r \\ \tfrac{1}{2}\pm\mu+r,\ \Delta(n, n/4-\nu+k-m-\tfrac{1}{2}) \end{matrix}\right) \qquad \ldots (3.1)$$

provided $R(1+\nu+n/4+m\pm m\pm n\mu)>0$, $R(m+\lambda+k+\nu-n/4+\frac{1}{2})<0$, $R(\rho+\lambda+\frac{n}{4})<0$ where $g(x)=0(x^{\rho})$ for large x, $R(\sigma\pm\mu+3/2)>0$ where $g(x)=0(x^{\sigma})$ for small x and the symbol $\Delta(n,a)$ denotes the set of

$$\frac{a}{n}, \frac{a+1}{n}, \frac{a+2}{n}, \ldots\ldots\ldots, \frac{a+n-1}{n}.$$

Proof : By hypothesis

$$\phi(p)=p\int_0^\infty (pt)^{m-\frac{1}{2}}\, e^{-pt/2}\, W_{k,m}\,(pt)\,\psi\,(t^n)\,dt$$

and ... (3.2)

$$p^{1-\nu/n}\,\psi\,(p)=p\int_0^\infty (2pt)^{-\frac{1}{4}}\, W_{\lambda,\mu}\,(2pt)\,g\,(t)\,dt$$

... (3.3)

Substituting the value of $\psi\,(t^n)$ from (3.3) in (3.2) we get

$$\phi(p)=p\int_0^\infty (pt)^{m-\frac{1}{2}}\, e^{-pt/2}\, W_{k,m}\,(pt)\,t^{\nu}\int_0^\infty (2xt^n)^{-\frac{1}{4}}\, W_{\lambda,\,\mu}\,(2xt^n)\; g(x).dx\;dt$$

$$=2^{-\frac{1}{4}}\,p\int_0^\infty x^{-\frac{1}{4}}\,g(x)dx\int_0^\infty (pt)^{m-\frac{1}{2}}\,e^{-pt/2}\,W_{k,m}\,(pt)W_{\lambda,\mu}\,(2xt^n)t^{\nu-n/4}\,dt$$

......, . (C)

$$=2^{-\frac{1}{4}}\,p\sum_{r=0}^{\infty}\frac{1}{\underline{|r}}\int_0^\infty x^{r-\frac{1}{4}}\,g(x)dx\int_0^\infty (pt)^{m-\frac{1}{2}}\,e^{-pt/2}\,W_{k,m}\,(pt)\,t^{\nu+nr-n/4}$$

$$e^{-xt^n}\quad W_{\lambda,\mu}\,(2xt^n)\,dt$$

...... (D)

Now making use of the relation (9., p. 403)

$$t^{-r\rho}\,e^{-zt^n/2}\,W_{\lambda,\mu}\,(zt^n)\,\frac{V_2}{k,m}\,p^{n\rho}\,(2\pi)^{(1-n)/2}\,n^{k+m-n\rho}\;\times$$

$$G^{2,\,2n}_{2n+1,\,2+n}\left(\frac{zn^n}{p^n}\;\middle|\;\begin{matrix}\Delta\,(n,\,n\rho),\,\Delta\,(n,\,n\rho-2m),\,1-\lambda\\ \frac{1}{2}\pm\mu,\,\Delta\,(n,\,n\rho+k-m-\frac{1}{2})\end{matrix}\right),$$

where $R(1-n\rho+m\pm m+n/2\pm n\mu)>0$ and $R(p)\geqslant 0$,

we obtain the theorem.

Now only thing remains is to justify the two inversions in the steps (C) and (D) in the proof.

The first inversion, the change of the order of integration in (C), is justified, under the conditions stated, by virtue of De la vallee Poussin's theorem (3., p. 504) as the asymptotic behaviour of Whittaker function is as in (2.4), since both x- and t- integrals converge absolutely and the double integral exists

The second inversion in the step (D) is valid since each term of the series

$$\frac{(xt^n)^r}{\lfloor r}$$

is continuous and

$$\sum_{=0}^{\infty} \left| \frac{(xt^n)^r}{\lfloor r} \right|$$

converges uniformly in the arbitrary large finite interval $0 < t < \alpha$

also $t^{m+v-n/4-\frac{1}{2}} \, e^{-pt/_2 - xt^n} \, W_{k,m}(pt) \, W_{\lambda,\mu}(xt^n)$

is bounded and integrable in the interval $0 < t < \alpha$,

Corollary 1. In the theorem if we take $\lambda = \pm \mu = \frac{1}{4}$ we get the theorem as :

If

$$\phi(p) \; \frac{V_2}{k,m} \; \psi(t^n)$$

and

$$p^{1-v/n} \psi(p) \doteqdot g(t)$$

then

$$\phi(p) = (2\pi)^{(1-n)/2} p^{-v} \, n^{v+m+k} \int_0^\infty G^{1,\,2n}_{2n,\,n+1}\left(\frac{n^n x}{p^n} \middle| \begin{matrix} \Delta(n, -v), \Delta, (n, -v-2m) \\ 0, \Delta(n, -v+k-m-\frac{1}{2}) \end{matrix} \right) g(x)\, dx \quad , \ldots (3.4)$$

provided R $(1+v+n/4 \pm n/4 + m \pm m) > 0$, R$(m+k+v-n/4) < 0$, R$(\rho+1) i < 0$, where $g(x) = O(x^{\rho})$ for large x, R$(\sigma+5/4) > 0$, where $g(n) = 0 \; (x^{\sigma})$ for small x and R$(p) > 0$.

For $n \leqslant 2$, the relation (3.4) can be written as

$$\phi(p) = \frac{(2\pi)^{\frac{1}{2}-n/2} \, n^{k+m+v} \; \Gamma(1+v/n) \ldots\ldots \Gamma(1/n+v/n) \; \Gamma(1+v/n+2m/n) \ldots\ldots \Gamma(1/n+v/n+2m/n)}{p^v \; \Gamma\left(1+\frac{v-k+m+\frac{1}{2}}{n}\right) \ldots\ldots \Gamma\left(1/n+\frac{v-k+m+\frac{1}{2}}{n}\right)}$$

$$\times \int_0^\infty {}_{2n}F_n \left[\begin{matrix} 1+v/n, \ldots\ldots, 1/n+\frac{v}{n}, 1+v/n+2m/n, \ldots, 1/n+v/n+2m/n \\ 1+\frac{v-k+m+\frac{1}{2}}{n}, \ldots\ldots, 1/n+\frac{v-k+m+\frac{1}{2}}{n} ; \; - \frac{n^n x}{p^n} \end{matrix} \right] g(x)\, dx \quad \ldots (3.5)$$

For $n=1$, we get the theorem given by RATHIE [8 ; p. 234]

Corollary 2. If

$$\phi(p) \doteqdot \psi(t^n)$$

and

$$p^{1-\nu/n}\, \psi(p) \frac{V_1}{\lambda,\mu}\; g(t)$$

then

$$\phi(p) = (2\pi)^{(1-n)/2}\, p^{-\nu}\, n^{\nu+\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r}{L^r} \int_0^{\infty} G^{2,\,n}_{n+1,\,2}\left(\frac{2n^n x}{p^n} \middle| \begin{matrix} \Delta(n,-\nu),\ \frac{3}{4}-\lambda+r \\ \frac{1}{4}\pm\mu+r \end{matrix} \right) g(x)\, dx \qquad \ldots (3.6)$$

provided $R(1+\nu+n/4\pm n/4)>0$, $R(\lambda-\nu-n/4+1)<0$, $R(\rho-\lambda+\frac{3}{4})<0$, where $g(x)=0(x^{\rho})$ for large x, $R(\sigma\pm\mu+3/2)>0$ where $g(x)=0(x^{\sigma})$ for small x and $R(p)>0$.

This corollary can be obtained by taking $k=\pm m=\frac{1}{4}$ in the theorem.

Further, if we take $n=1$ and use the result (2.9), we get the theorem due to BOSE (2., p. 19).

Example : If we take (1., p. 13)

$$g(t) = t^{\sigma}\, e^{-qt},$$

then

$$p^{1-\nu/n}\, \psi(p) = \frac{V_1}{\lambda,\mu}\; g(t) = \frac{\Gamma(\sigma\pm\mu+5/4)}{2(2p)^{\sigma}\, \Gamma(\sigma-\lambda+7/4)}\; {}_2F_1\left[\begin{matrix} \sigma\pm\mu+5/4 \\ \sigma-\lambda+7/4 \end{matrix} ;\ \frac{1}{2}-\frac{q}{2p} \right] \qquad \ldots (3.7)$$

where $R(\sigma\pm\mu+5/4)>0$, $R(p) > R(p_0) > 0$ and $|p| > |q|$

Putting the value of $g(t)$ in the relation (3.1) and on solving the integral, we get

$$\phi(p) = \frac{(2\pi)^{(1-n)/2}\, p^{n/4-\nu}\, q^{-\sigma-\frac{3}{4}}}{2^{\frac{1}{4}}\, n^{n/4-k-m-\nu}} \times$$

$$\times \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r}{L^r}\, G^{2,\,2n+1}_{2n+2,\,n+2}\left(\frac{2n^n}{p^n q} \middle| \begin{matrix} \frac{1}{4}-\sigma,\ \Delta(n,\, n/4-\nu),\ \Delta(n,\, n/4-\nu-2m),\ 1-\lambda+r \\ \frac{1}{2}\pm\mu+r,\ \Delta(n,\, n/4-\nu+k-m-\frac{1}{2}) \end{matrix} \right), \qquad \ldots\ldots (3.8)$$

where $R(p)>0$ and $R(1+\nu+n/4+m\pm m\pm n\mu)>0$.

Now using the value of $\psi(p)$ in the relation (3.2) and equating this integral with R. H. S. of the relation (3.8) and on adjusting the parameters, we get

$$\int_0^\infty t^{v-1} e^{-pt/2} W_{k,m}(pt)\, {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \tfrac{1}{2} - \frac{q}{2} t^{-n}\right] dt$$

$$= \frac{(2\pi)^{(1-n)/2}\, p^{-v+n/2-n(a+b)/2}\, (q)^{(1-a-b)/2}}{\Gamma(a)\,\Gamma(b)\, n^{n/2-k-v+\frac{1}{2}-n(a+b)/2}} \frac{\Gamma(c)}{2^{-n(a+b)/2+5n/4-\frac{5}{4}}} \times$$

$$\times \sum_{r=0}^{\infty} \frac{(\frac{1}{2})^r}{\lfloor r} G^{2,\,2n+1}_{2n+2,\,n+2}\left(\frac{2n^n}{p^n q}\middle|\begin{matrix}(3-a-b)/2,\ \triangle(n,\ n/2-v-n(a+b)/2+m+\frac{1}{2}),\ \triangle(n, -n(a+b)/2+n/2-m-v+\frac{1}{2}),\ \frac{1}{2}-a/2-b/2+c+r \\ \frac{1}{2}\pm(a-b)/2+r,\ \triangle(n,\ (n/2-v+k-n/2\,(a+b))\end{matrix}\right) \quad \ldots (3.9)$$

where $R(a)>0$, $R(b)>0$, $R(p) > R(p_0)>0$, $|p| > |q|$ and $R(\frac{1}{2}+v+n/2\,(a+b) \pm m \pm n/2\,(a-b))>0$.

In this relation if we take $b=c$, we get

$$\int_0^\infty t^{v+an-1} e^{-pt} W_{k,m}(pt)\,(t^n - q)^{-a}\, dt$$

$$= \frac{(2\pi)^{(1-n)/2}\, p^{-v+n/2-na/2}\, q^{(1-a)/2}}{\Gamma(a)\, 2^{5n/4-5/4+a/2-na/2}\, n^{n/2-k-v+\frac{1}{2}-na/2}} \times$$

$$G^{1,\,2n+1}_{2n+1,\,n+1}\left(\frac{2n^n}{p^n q}\middle|\begin{matrix}3/2-a/2,\ \triangle(n,\ n/2-v\pm m+\frac{1}{2}-na/2), \\ \frac{1}{2}+a/2,\ \triangle(n,\ n/2+k-v-na/2)\end{matrix}\right) \quad \ldots (3.10)$$

where $R(a)>0$, $R(p) > R(p_0)>0$, $|p| > |q|$ and $R(\frac{1}{2}+v\pm m+na/2\pm+nb/2)>0$.

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SOME PROPERTIES OF A GENERALIZATION OF LOMMEL AND MAITLAND TRANSFORMS

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ABSTRACT

In this paper some functional relations satisfied by the generalized transform

$$\phi(x) = \int_0^{\infty} (xy)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy) f(y)\, dy$$

have been established. Transforms of higher derivatives of the functions $f(x)$ have also been obtained. The results have been illustrated by means of a few examples.

1. In a series of papers* I have studied the properties of the function

$$J^{\mu}_{\nu,\lambda}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{\nu+2\lambda+2r}}{\Gamma(1+\lambda+r)\,\Gamma(1+\lambda+\nu+\mu r)}, \quad (\mu > 0),$$

which reduces to Lommel's function (Watson, 1958) for $\mu = 1$ and to Maitland's function (Wright, 1933) for $\lambda = 0$.

Also, with the help of this function I have given a generalization of the Lommel-transform (Hardy, 1925),

$$g(y) = \int_0^{\infty} (xy)\; F_{\nu}(xy) f(x)\, dx, \tag{1.1}$$

where

$$F_{\nu}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{\nu+2\lambda+2r}}{\Gamma(1+\lambda+r)\,\Gamma(1+\lambda+\nu+r)} = \frac{2^{2-\nu-2\lambda}}{\Gamma(\lambda)\,\Gamma(\nu+\lambda)}\, s_{\nu+2\lambda-1,\,\nu}(x),$$

in the form

$$g(y) = \int_0^{\infty} (xy)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy) f(x)\, dx, \tag{1.2}$$

and obtained an inversion formula for this generalization, when λ is a non-negative integer in the form

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$$f(x) = 1/\mu \ (-1)^{\lambda} \ \sqrt{2} \int_{0}^{\infty} (xy)^{2/\mu(\nu+\lambda+1)-\nu-2\lambda-3/2} \, J^{1/\mu}_{(\nu+\lambda+1)/\mu-\lambda-1} \left[\left(\frac{xy}{2} \right)^{2/\mu} \right] \times g(y) \, dy,$$

under certain appropriate conditions. It may be noted that (1.2) reduces to (1.1) for $\mu = 1$ and to the generalized Hankel-transform (Agarwal, 1950) for $\lambda = o$, which we call as the Maitland-transform.

In this paper I have obtained some general formulae for this generalized Lommel transform. Some of these formulae are similar to those of the Hankel transform and the generalized Hankel transform, studied by Kumar (1961), while the analogues of others do not exist in the above Hankel transforms theory. I have also illustrated their applications by means of some examples.

We shall denote $g(y)$, the generalized Lommel transform of $f(x)$, by the symbol.

$$g(y; \mu, \nu, \lambda,) = J(f(x)).$$

To begin with, we have (Wright, 1935), as $x \rightarrow \infty$

$$J^{\mu}_{\nu,\lambda}(x) = O\left[x^{\nu+2\lambda-2k(\nu+2\lambda+1/2)} \, exp \left\{ (\mu x^2/4)^k \, \frac{cos(\pi k)}{\mu k} \right\} + \frac{x^{\nu+2\lambda-2}}{\Gamma(\lambda)\Gamma(\nu+\lambda+1-\mu)} \right], \ (o < \mu \leqslant 1),$$

$$= O\left[x^{\nu+2\lambda-2k(\nu+2\lambda+1/2)} \, exp \left\{ (\mu x^2/4)^k \, \frac{cos \, \pi k}{\mu k} \right\} \right], \ (\mu > 1),$$

where $K = \dfrac{1}{(1+\mu)}$.

2. From (1.2) we easily get

(1) $$1/a/g\left(y/a; \mu, \nu, \lambda\right) = J\left(f(ax)\right).$$

(2) $$\frac{y}{2(\nu+\lambda-\mu\lambda)}\left[g(y; \mu, \nu-1, \lambda) - \mu g(y; \mu, \nu+1, \lambda-1)\right] = J\left(f(x)/x\right),$$

on using the formula (Pathak, 1963)

$$x \, J^{\mu}_{\nu-1,\lambda}(x) = \mu x \, J^{\mu}_{\nu+1,\lambda-1}(x) + 2(\nu+\lambda-\mu\lambda) \, J^{\mu}_{\nu,\lambda}(x). \quad \text{(A)}$$

(3) $y^{\nu+1/2} \left(1/y \dfrac{d}{dy} \right)^m \left[g\,(y;\mu,\nu-m,\lambda+m)\; y^{-\nu-1/2+m} \right]$

$$= \mathrm{J}\left(x^m f(x) \right),\; m=0,1,2,\ldots,$$

on using the formula (Pathak, 1963) :

$$\left(1/x \frac{d}{dx} \right)^m \left(x^{-\nu} J^{\mu}_{\nu,\lambda}(x) \right) = x^{-\nu-m} J^{\mu}_{\nu+m,\lambda-m}(x),\; (m=0,1,2,\ldots). \tag{B}$$

EXAMPLE : Let $\qquad f(x) = x^{\frac{1}{2}-\nu}\, e^{-x^2}$,

Then, by tem/by term integration

$$g\,(y;\mu,\nu,\lambda) = y^{\frac{1}{2}} \int_0^{\infty} x^{1-\nu}\, e^{-x^2} J^{\mu}_{\nu,\lambda}(xy)\, dx$$

$$= y^{\nu+2\lambda+1/2}\; 2^{-\nu-2\lambda-1}\; E_{\mu,1+\nu+\lambda}\,(-y^2/4), \tag{2.1}$$

where $\mu > O$, $R(\lambda) > -1$ and $E\alpha,\beta$ is the generalized Mittag — Leffler function (Erdélyi, 1955).

Therefore, from (3)

$$\int_0^{\infty} x^{m+1-\nu}\, e^{-x^2} J^{\mu}_{\nu,\lambda}(xy)\, dx$$

$$= 2^{-\nu-m-2\lambda-1}\, y^{\nu} \left(1/y \frac{d}{dy} \right)^m \left[y^{2\lambda+2m}\, E_{\mu,1+\nu+\lambda}\,(-y^2/4) \right], \tag{2.2}$$

where $\quad \mu > o$, $R(\lambda) > -(m/2+1)$ and $m = 0,1,2\ldots.$

(4) $\quad y^{1/2+\nu} \displaystyle\int_0^{y} \eta^{-1/2-\nu}\, g(\eta;\mu,\nu+1,\lambda-1)\, d\eta = \mathrm{J}\,(f(x)/x),\; (R(\lambda) > o).$

This can be easily proved by using the formula (Pathak, 1963) .

$$\int_0^{x} x^{-\nu} J^{\mu}_{\nu+1,\lambda-1}(x)\, dx = x^{-\nu} J^{\mu}_{\nu,\lambda}(x), \quad (R(\lambda) > o).$$

(5) $\quad -y^{\frac{1}{2}+\nu} \displaystyle\int_y^{\infty} \eta^{-\frac{1}{2}-\nu}\, g\,(\eta;\mu,\, \nu+1,\lambda-1)\, d\eta = \mathrm{J}(f(x)/x),$

where $0 < \mu \leqslant 1$ and $R(\lambda) < 1$ when $R(\lambda)$, $R(\lambda + \nu + 1 - \mu) \neq 0, -1, -2, \dots$,

with an additional condition $R(\nu) > -\frac{1}{2}$ in case $\mu = 1$.

This can be proved by using the same formula as employed in (4).

EXAMPLE :

Let $f(x) = x^{-\nu+1/2} e^{-x^4/4} D_\lambda(x^2)$ and $\mu = \frac{1}{2}$.

Then, by term by term integration

$$g(y;\tfrac{1}{2},\nu,\lambda) = y^{1/2} \int_0^\infty x^{1-\nu} e^{-x^4/4} D_\lambda(x^2) J_{\nu,\lambda}^{1/2}(xy)\, dx$$

$$= \sqrt{\pi}\, 2^{1/2(3\nu+\lambda-3)} y^{-\nu+1/2} \left[I_{\nu+\lambda}\left(\frac{y^2}{2\sqrt{2}}\right) - L_{\nu+\lambda}\left(\frac{y^2}{2\sqrt{2}}\right) \right], \quad (2.3a)$$

where $R(\lambda) > -1$.

Therefore, from (4)

$$\int_0^y \eta^{-2\nu-1} \left[I_{\nu+\lambda}\left(\frac{\eta^2}{2\sqrt{2}}\right) - L_{\nu+\lambda}\left(\frac{\eta^2}{2\sqrt{2}}\right) \right] d\eta$$

$$= y^{2\lambda}\, 2^{-\frac{1}{2}(5\nu+5\lambda+3/2)} \sum_{r=0}^{\infty} \frac{\left(-y^2/(4\sqrt{2})\right)^r \Gamma(\lambda+1/2+r)}{\Gamma(1+\lambda+r)\, \Gamma(1+\lambda+\nu+r/2)\, \Gamma(3/4+r/2)}, \quad (2.3)$$

where $R(\lambda) > 0$.

In particular, for $\nu = 1/4 - \lambda$, we have

$$\int_0^y \eta^{2\lambda-5/2} \left[I_{1/4}\left(\frac{\eta^2}{2\sqrt{2}}\right) - L_{1/4}\left(\frac{\eta^2}{2\sqrt{2}}\right) \right] d\eta$$

$$= \frac{2^{1/8}}{\eta} \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)}\, y^{2\lambda}\, {}_1F_2\left(\begin{matrix} \lambda+1/2 \\ \lambda+1, 3/2 \end{matrix}; -\frac{y^2}{2\sqrt{2}} \right),$$

where $R(\lambda) > 0$.

Also, from (5), we have

$$\int_y^\infty \eta^{-(2\nu+1)} \left[I_{\nu+\lambda}\left(\frac{\eta^2}{2\sqrt{2}}\right) - L_{\nu+\lambda}\left(\frac{\eta^2}{2\sqrt{2}}\right) \right] a\eta$$

$$= -y^{2\lambda}\, 2^{-1/2(5\nu+5\lambda+3/2)} \sum_{r=0}^{\infty} \frac{\left(-y^2/(4\sqrt{2})\right)^r \Gamma(\lambda+1/2+r)}{\Gamma(1+\lambda+r)\,\Gamma(1+\lambda+\nu+r/2)\,\Gamma(3/4+r/2)}, \tag{2.4}$$

where $-\frac{1}{2} < R(\lambda) < 1.$

And in particular, for $\nu = 1/4 - \lambda$, (2.4) gives

$$\int_y^{\infty} \eta^{2\lambda-3/2} \left[I_{1/4}\left(\frac{\eta^2}{2\sqrt{2}}\right) - L_{1/4}\left(\frac{\eta^2}{2\sqrt{2}}\right)\right] d\eta$$

$$= -\frac{2^{1/8}}{\pi} \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} y^{2\lambda}\ {}_1F_2\left(\begin{matrix}\lambda+1/2\\ \lambda+,3/2\end{matrix};-\frac{y^2}{2\sqrt{2}}\right), \tag{2.4.}$$

where $-\frac{1}{2} < R(\lambda) < 1.$

(6) $$\frac{y^{-2/\mu(\nu+\lambda)+\nu+2\lambda+1/2}}{\mu 2^{\sigma-1}\,\Gamma(\sigma)} \int_0^y \eta^{2/\mu(\nu+\lambda-\sigma+)-\nu+\sigma-2\lambda-3/2} \times$$

$$\times (y^{2/\mu} - \eta^{2/\mu})^{\sigma-1} g(\eta;\mu,\nu-\sigma,\lambda)\, d\eta$$

$$= J\left(\frac{f(x)}{x^{\sigma}}\right), \quad \left(R(\nu+\lambda)+1 > R(\sigma) > O\right).$$

This can be easily proved by substituting the value of $g(\eta;\mu,\nu-\sigma,\lambda)$ in the integrand on the left and using the formula (Pathak, 1963 A)

$$\eta^{-\sigma} J^{\mu}_{\nu,\lambda}(x\eta) = \frac{x^{-2/\mu(\nu+\lambda)+\nu+2\lambda}}{\mu\ 2^{\sigma-1}\ \Gamma(\sigma)} \times$$

$$\times \int_0^x y^{2/\mu(\nu+\lambda-\sigma+1)-\nu-2\lambda+\sigma-1}\left(x^{2/\mu} - y^{2/\mu}\right)^{\sigma-1} J^{\mu}_{\nu-\sigma,\lambda}(y\eta)\, dy,$$

where $R(\nu+\lambda)+1 > R(\sigma) \geq O$ and $\mu > O$.

(7) $$\frac{(-1)^p\ y^{\nu+1/2}}{2^{p-1}\ (p-)!} \int_y^{\infty} \eta^{-\nu-p+1/2} (\eta^2-y^2)^{p-1} g\,(\eta;\mu,\nu+p,\lambda-p)\,d\eta$$

$$= J\left(f(x)/x^{p}\right),$$

where $O < \mu \leqslant 1$, p is a positive integer and $R(\lambda) < 1$ when $R(\lambda - p)$,

$R(\gamma+\nu-\mu+1) \neq 0, -1, -2, \ldots$, with an additional condition $R(\nu) > p - 1/2$ when $\mu = 1$.

The proof is similar to (6). Here we use the formula (Pathak, 1963B)

$$\eta^{-p} J^{\mu}_{\nu,\lambda}(\eta x) = \frac{(-1)^p x^{\nu}}{2^{p-1}(p-1)!} \int_x^{\infty} y^{-\nu-p+1} (y^2-x^2)^{p-1} J^{\mu}_{\nu+p,\lambda-p}(\eta y)dy,$$

where $0 < \mu \leqslant 1$, p is a positive integer and $R(\lambda) < 1$ when $R(\lambda-p)$, $R(\lambda+\nu-\mu+1) \neq 0, -1, -2, \ldots$, with an additional condition $R(\nu) > p - \frac{1}{2}$ in case $\mu = 1$.

EXAMPLE. Taking $f(x) = x^{1/2-\nu} e^{-x^2}$,

it is easy to show from (6), that

$$\int_0^y \eta^{2/\mu(\nu+\lambda-\sigma+1)-1} \left(y^{2/\mu} - \eta^{2/\mu}\right)^{\sigma-1} E_{\mu,1+\nu-\sigma+\lambda}\left(-\eta^2/4\right) d\eta$$

$$= \frac{\mu}{2} \Gamma(\sigma)\, y^{2/\mu(\nu+\lambda)} \sum_{r=0}^{\infty} \frac{(-y^2/4)^r\ \Gamma(\lambda-\sigma/2+1+r)}{\Gamma(1+\lambda+r)\ \Gamma(1+\lambda+\nu+\mu r)}, \tag{2.5}$$

where $\mu > 0$, $R(\lambda+\nu)+1 > R(\sigma) > 0$ and $R(\lambda-\sigma/2) > -1$.

Also, from (7) we have

$$\int_y^{\infty} \eta^{2\lambda-2p+1} (\eta^2-y^2)^{p-1} E_{\mu,1+\nu+\lambda}\left(-\frac{\eta^2}{4}\right) d\eta$$

$$= \frac{1}{2}(-1)^p (p-1)!\, y^{2\lambda} \sum_{r=0}^{\infty} \frac{(-y^2/4)^r\ \Gamma(\lambda-p/2+1+r)}{\Gamma(1+\lambda+r)\ \Gamma(1+\lambda+\nu+\mu r)}, \tag{2.6}$$

where $0 < \mu < 2$, $p/2-1 < R(\lambda) < 1$ and $p = 1, 2, 3$.

3. GENERALIZED LOMMEL TRANSFORMS OF THE DERIVATIVES OF A FUNCTION :

(1) $$-\frac{(1+2\nu)}{4(\nu+\lambda-\mu\lambda)}\, y\, g(y;\mu,\nu-1,\lambda) + \left\{\frac{\mu(1+2\nu)}{4(\nu+\lambda-\mu\lambda)} - 1\right\} y\, g(y;\mu,\nu+1,\lambda-1)$$

$$= J\left(\frac{d}{dx} f(x)\right), \tag{3.1}$$

provided $x^{1/2} J^{\mu}_{\nu,\lambda}(xy) f(x)$ tends to zero both as $x \to 0$ and $x \to \infty$.

For, by integrating by parts

$$g'(y;\mu,\nu,\lambda) = \int_0^\infty (xy)^{1/2} J^{\mu}_{\nu,\lambda}(xy) \frac{d}{dx} f(x)\, dx$$

$$= -y^{1/2} \int_0^\infty f(x) \frac{d}{dx}\left[x^{1/2} J^{\mu}_{\nu,\lambda}(xy) \right] dx,$$

provided $x^{1/2} J^{\mu}_{\nu,\lambda}(xy) f(x)$ tends to zero as x tends either to zero or to infinity.

Now, using the formula (Pathak, 1965):

$$x J'^{\mu}_{\nu,\lambda}(x) = \nu J^{\mu}_{\nu,\lambda}(x) + x J^{\mu}_{\nu+1,\lambda-1}(x),$$

we have

$$g'(y;\mu,\nu,\lambda) = -(\nu+{}^1/_2)\int_0^\infty (xy)^{1/2} J^{\mu}_{\nu,\lambda}(xy) f(x)/x \quad dx$$

$$-y\int_0^\infty (xy)^{1/2} J^{\mu}_{\nu+1,\lambda-1}(xy) f(x)\, dx,$$

which, by applying the formula (A) proves the proposition.

The formulae for the generalized Lommel-transforms of higher derivatives of the function $f(x)$ may be obtained by repeated applications of the above relation (1).

EXAMPLE : Taking $f(x) = x^{p-3/2} e^{-1/2 x^2} W_{\rho,\sigma}(x^2)$,

we easily see, from (1), that

$$\int_0^\infty (xy)^{1/2} J^{\mu}_{\nu,\lambda}(xy) \frac{d}{dx}\left(x^{p-3/2} e^{-x^2/2} W_{\rho,\sigma}(x^2)\right), dx.$$

$$= -\frac{(1+2\nu)}{(\nu+\lambda-\mu\lambda)} \frac{\Gamma_*\left((\nu+p)/2 + \lambda \pm \sigma\right)}{\Gamma(1+\lambda)\,\Gamma(\nu+\lambda)\,\Gamma(\nu/2+p/2+{}^1/_2+\lambda-\rho)} y^{\nu+2\lambda+1/2}$$

$$2^{-\nu-2\lambda-2} \times$$

$$\times\ {}_3F_{\mu+2}\left(\begin{matrix}1,\ \nu/_2+p/_2+\lambda+\sigma,\ \nu/_2+p/2+\lambda-\sigma & ;\ -y^2/_4\mu^{-\mu}\\ 1+\lambda,\ \nu/_2+p/_2+{}^1/_2+\lambda-\rho,\ (\nu+\lambda)/\mu,\ (\nu+\lambda+1)/\mu,\ \ldots,\ (\nu+\lambda+\mu-1)/\mu & \end{matrix}\right)$$

$$+\left(\frac{\mu(1+2\nu)}{4(\nu+\lambda-\mu\lambda)}-1\right)\frac{\Gamma_\star(\nu/_2+p/_2+\lambda\pm\sigma)}{\Gamma(\lambda)\ \Gamma(1+\nu+\lambda)\ \Gamma(\nu/_2+p/_2+{}^1/_2+\lambda+\rho)}$$

$$\times\ y^{\nu+2\lambda+{}^1/_2}\ 2^{-\nu-2\lambda}\ {}_3F_{\mu-2}\left(\begin{matrix}1,\ \nu/_2+p/_2+\lambda+\sigma,\nu/_2+p/_2+\lambda-\sigma\ ;-y^2/_4\mu^{-\mu}\\ \lambda,\ \nu/_2+p/_2+1+\lambda+\rho,(1+\nu+\lambda)/\mu,\ (2+\nu+\lambda)/\mu,\\ \ldots,(\mu+\nu+\lambda)/\mu\end{matrix}\right)$$

provided that $R\ (\nu+2\lambda\pm2\sigma)>-p$ and μ is a positive integer.

(2) $(-1)^m y^m g(y;\mu,\nu+m,\lambda-m)$

$$=J\left[x^{\frac{1}{2}-\nu}\left(\frac{1}{x}\ \frac{d}{dx}\right)^m\left(x^{\nu+m-{}^1/_2}\ f(x)\right)\right],\ m=0,1,2,\ldots,$$

provided that $\left(\frac{1}{x}\ \frac{d}{dx}\right)^r\left[x^{-\nu}\ J^{\mu}_{\nu,\lambda}\ (xy)\right]\left(\frac{1}{x}\ \frac{d}{dx}\right)^{m-r-1}$

$$\left[x^{\nu+m-\frac{1}{2}}\ f(x)\right],\ r=0,\ 1,\ 2,\ldots,\ m-1,$$

vanishes both at O and at ∞.

This can be easily proved by induction method on using the formula (B).

EXAMPLE : Let $f(x)=x^{-\nu+{}^1/_2}\ exp(-x^4/_4)\ D_\lambda\ (x^2)$ and $\mu={}^1/_2$.

Then, from (2.3a)

$$g\,(y;\mu,\nu,\lambda)=\sqrt{\pi}\ 2^{{}^1/_2\,(3\nu+\lambda-3)}\ y^{-\nu+{}^1/_2}\left[I_{\nu+\lambda}\left(\frac{y^2}{2\sqrt{2}}\right)-L_{\nu+\lambda}\left(\frac{y^2}{2\sqrt{2}}\right)\right],$$

where $R(\lambda)>-1$.

Therefore, from (2)

$$\int_0^\infty x^{1-\nu}\ J^{\frac{1}{2}}_{\nu,\lambda}\ (x\ y)\left(\frac{1}{x}\ \frac{d}{dx}\right)^m\left[x^m\ exp(-x^4/_4)\ D_\lambda\ (x^2)\right]dx$$

$$=(-1)^m\ \sqrt{\pi}\ 2^{{}^1/_2\,(3\nu+2m+\lambda-3)}\ y^{-\nu}\left[I_{\nu+\lambda}\left(\frac{y}{2\sqrt{2}}\right)-L_{\nu+\lambda}\left(\frac{y}{2\sqrt{2}}\right)\right],$$

where $R(\lambda) > -1$ and m is a nonnegative integer.

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EFFECT OF SOME HERBICIDES ON SOIL MICRO-ORGANISMS

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ABSTRACT

The effect of six herbicides in two different doses on microbiological population was studied. In general, all the herbicides, except planotox, did not show any adverse effect on the bacterial and fungal population of soil. Planotox proved to be toxic to the growth of fungi.

The microbial population of soil is dependent upon the maintenance of proper soil environment. The application of herbicides may prove beneficial or harmful to the growth of soil microbes. The micro-organisms use all types of organic matter including organic herbicides. Certain herbicides may remain toxic in the soil if the soil is cold, dry and poorly aerated. It is, therefore, obvious to determine the extent to which different herbicides being used in different quantities affect the growth of the soil micro-organisms and various other allied cyclic processes.

A considerable amount of research work has been done on the use of different herbicides. Pokorny (1941), Zimmermann and Hitchcock (1942) and Marth and Mitchell (1944) have contributed a lot by reviewing the effect of 2.4-D on soil microflora. Smith et al (1946) demonstrated that concentrations of 2,4-D ranging from 0.5 to 500 p, p. m, added to a sandy soil of good fertility did not appreciably influence the microbial population. Newman (1947) found that 2,4-D, McPA (4-chloro-2 methylphenoxyacetic acid), and 2,4,5-T (2,4,5-trichlorophenoxy acetic acid) were more fungitoxic under acid conditions. Kratochvil (1951) had shown that rates of 2,4,5-T upto 16 lbs, per acre have no effect on the carbon dioxide evolution of treated soils. Wrobel (1952) found that herbicidal concentrations of 2,4-D as applied in agriculture have no detrimental effect on Rhizobium trifolii and Rhizobium-lupini. Hoover and Colmer (1953) showed that applications of the triethanolamine salt of 2,4,-D, upto 250 fold regular field applications had no appreciable effect upon the number of fungi, actinomycetes and Bacteria in a clay soil. Fletcher (1956) and Fletcher et al (1956) have shown that concentration 2,4-DB, 4 (2,4, Dichlorophenoxy butyric acid) and McPB, 4(4-Chloro-2 methylphenoxy butyric acid) upto 500 p. p. m have no adverse effect on the growth of Rhizobium trifolii. The effect of TCA (Trichloroacetic acid) on soil organisms follows the pattern for many other herbicides, having an initial inhibitive effect at accepted rates for weed control (Kratochvil 1951) followed by a recovery of the soil population (Colmer 1953b, 1954, Hoover and Colmer, 1953, Paixao and Dobereiner 1956) found that over 1000 p. p. m, TCA was required to produce marked decreases in the soil population. Fungi were even more resistant, some species resisting concentrations as high as 40,000 p. p. m. The present investigation was undertaken to see the effect of herbicides on soil micro-organisms.

EXPERIMENTAL

Six herbicides were selected for study and the experiments were conducted under controlled laboratory conditions.

Herbicides used with recommended doses

(1) Simazine :— 2 Chloro-4,6-Bis (Ethylamino)-S-Triazine. A. E. 46%, dose 2 and 5 lbs per acre.

(2) Tropotox :— (Sodium salt of McPB) A. E. 40%, dose 4 and 8 pints per acre.

(3) Spontox (2,4-D and 2,4, 5-T) A, E. 70%, dose 2.5 and 5 pints per acre.

(4) NaTA:—Sodium salt of trichloroacetic acid), dose 40 and 150 lbs per acre.

(5) Fernoxone (sodium salt of 2,4-D), A. E. 80%, dose 2 and 5 lbs per acre.

(6) Planotox :— (Butoxy-ethyl-ester of 2,4-D), A. E. 70%, dose 0.5 and 1.5 lbs per acre.

The soil for the study was collected from the Students, Instructional Farm, Govt. Agricultural College, Kanpur. The respective doses of the herbicides were mixed in the soil properly inorder to maintain uniform distribution of the herbicides in the soil. The moisture content of the soil was regulated from time to time and was maintained according to the optimum field conditions. Thorough stirring of the soil was done in order to break lump formation, if any and to provide sufficient aeration. The sampling of the soil was done after an interval of every 25 days. The total bacteria and fungi were counted by plate method as described by Waksman (1928).

RESULTS AND DISCUSSION

Data incorporated in table no. 1 indicate that different herbicides in both lower and higher doses as used in experimental studies do not have any adverse effect on the bacterial population of soil. Rather it was noted that under controlled laboratory conditions, there was in general an increase in the number of total bacteria. This falls in line with the observations of Goarin and Armand (1957) who showed that sodium salt of 2,4-D,McPA and triethanolamine salt of 2,4-D do not adversely affect aerobic cellulose decomposing bacteria, nitrifiers, azotobacter and clostridium. Verona (1948) has found that herbicides of the type 2,4-D applied at customary rates showed no diminution in the number of total bacteria. It was also observed that the higher doses of herbicides were less beneficial than the lower ones. The higher dose of planotox has been observed slightly toxic to the total bacterial population. The toxic effect of planotox may be attributed to the fact that chemically it is butoxy-ethyl-ester of 2,4-D, and the esters in general are more toxic (Klingman, 1961).

The data in table no. 2, reveal the effect of herbicides on the fungal population of soil. It was observed that the herbicides like Simazine and Tropotox in both doses accelerated the growth of fungi. Spontox, NaTA and Feronxone were found to be without any adverse effect. Planotox was the only herbicide which inhibited the growth of fungi. The effect of NaTA (sodium salt of trichloroacetic acid) follows the same pattern having an initial inhibitive effect followed by recovery of the soil population (Kratochvil, 1951, Colmer 1953 b, 1954). Hoover and Colmer (1953) have found that rates of trichloroacetic acid in excess of those used in field practice had no adverse effect upon fungi :

SUMMARY

Effect of six different herbicides as Simazine, Tropotox, Spontox, NaTa, Fernoxone and Planotox was studied on the microbial population of soil. All the herbicides, except Planotox, were found to be without any adverse effect on bacterial and fungal population of soil. Planotox was observed to be slightly toxic to the growth of fungi.

TABLE 1

TOTAL BACTERIA (Lacs per gm of soil)

Days	CONTROL	SIMAZINE		TROPOTOX		SPONTOX		NaTA		Fernox-one		PLANOTOX	
		Lower dose	Higher dose	Lower dose	Higher dose	Lower dose	Higher dose	Lower dose	Higer dose	Lower dose	Higher dose	Lower dose	Higher dose
0	6·25	6·25	6·25	6·25	6·25	6·25	6·25	6·25	6·25	6·25	6·25	6·25	6·25
25	6·00	10·00	9·63	10·25	8·75	8·75	6·25	6·50	6·00	5·05	7·00	6·45	5·00
50	5·50	10·63	10·00	8·75	9·13	7·50	6·88	9·38	7·50	10·00	11·00	10·25	7·50
75	5·60	10·40	4·00	7·20	5·20	4·00	4·00	6·80	5·60	5·00	5·50	4·00	3·80
Average	5·84	9·32	7·47	8·11	7·33	6·63	5·84	7·23	6·34	6·57	7·44	6·74	5·60

TABLE 2

TOTAL FUNGI (Thousands per gm of soil)

Days	CONTROL	SIMAZINE		TROPOTOX		SPONTOX		NaTA		Fernoxone		PLANOTOX	
		Lower dose	Higher dose	Lower dose	Higher dose	Lower dose	Higher dose	Lower dose	Higher dose	Lower dose	Higher dose	Lower dose	Higher dose
0	13·75	13·75	13·75	13·75	13·75	13·75	13·75	13·75	13·75	13·75	13·75	13·75	13·75
25	8·75	17·50	13·75	13·60	16·25	5·00	6·25	8·75	6·25	7·50	5·00	6·25	5·00
50	7·50	7·50	6·25	6·25	6·25	8·76	10·00	7·50	10·00	10·00	12·50	5·00	6·25
75	10·00	11·25	8·75	12·00	10·60	12·00	9·30	8·75	12·50	6·25	6·25	7·50	10·00
Average ...	10 00	12·50	10·62	11·40	11·71	9·88	9·82	9·69	10·62	9·37	9·38	8·13	8·75

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STUDIES ON PHOSPHORUS STATUS IN SOME ALKALI AND ADJOINING SOILS OF UTTAR PRADESH

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ABSTRACT

Three alkali as well as their adjoining soil profile samples were analysed for total, organic and inorganic phosphorus contents. The average values of total and inorganic phosphorus were found to be higher in adjoining soil profiles than in the alkali soil profiles. Organic phosphorus content was nearly similar in both the types of soil profiles.

Organic and inorganic phosphorus contents were significantly (positive) correlated with total phosphorus content in both the typs of soil profiles. Significant positive correlations were found between organic phosphorus and soil pH in both the types of soil profiles, but inorganic phosphorus showed significant positive correlation with soil pH only in the alkali soil profiles. Organic phosphorus and total sulphur were significantly correlated (positive) in both the types of soil profiles.

Soil phosphorus occurs mainly in two forms : (1) Organic and (2) Inorganic. The latter form is responsible for the main supply of available phosphorus and occurs in soil in combination with iron, aluminium, calcium and magnesium. The role of organic phosphorus in plant nutrition is thought to be of conspicuous nature. Its importance to crop plants, specially in mineral tropical soils as well as in organic matter rich soils of temperate countries, is not clearly understood (Goel and Agarwal, 1960). The forms of soil phosphorus are greatly influenced by other soil components such as pH, $CaCO_3$, carbon, nitrogen, sulphur etc., as seen in the subsequent discussion.

In the present investigation a depthwise determination of total, organic and inorganic phosphorus was made for a comparative study of the alkali and adjoining soil types in the light of some soil factors related to them.

EXPERIMENTAL

Samples from three alkali soil profiles as well as from their adjoining cultivable fields were collected from a few districts of Uttar Pradesh, *viz.*, Ballia, Jaunpur and Varanasi. Genetically, all the soils are Gangetic alluvium affected by salinity and alkalinity. All the soil samples were found to be on the alkaline side with pH ranging from 7.2 to 9.4.

Total phosphorus of the soil was extracted by sodium carbonate fusion method (Muir, 1952). Organic phosphorus was measured as the increase in inorganic phosphorus extracted by 0.2N $-H_2SO_4$, after ignition of the soil at 550°C for an hour in a muffle furnace (Saunders and Williams, 1955). Inorganic phosphorus was obtained by substracting organic fraction from the total one. All the phosphorus determinations were made colorimetrically following the method of Truog and Meyer (1929). Sulphur, nitrogen and organic carbon were determined as described in an earlier paper (Singh and Singh, 1966).

RESULTS AND DISCUSSIONS

The average values of total, inorganic and organic phosphorus contents in alkali soil profiles are 0.01903, 0.01473 and 0.0043 per cent respectively. The corresponding figures in adjoining soil profiles are 0.02588, 0.02120 and 0.0041 percent respectively. The average values of total and inorganic phosphorus are found to be higher in adjoining than in alkali soil profiles, while the values for organic phosphorus are closely similar in both the types of soil profiles. They show no definite trend of distribution with the depth of the soil profiles. Total phosphorus content of the soils differs widely from those reported by Kanwar and Grewal (1959) in Punjab soils and Goel and Agarwal (1959) in soils of Kanpur.

In the present investigation ratio of organic to total phosphorus (Table 1) ranges from 0.10 to 0.49 and from 0.05 to 0.34 in alkali and adjoining soil profiles respectively. The results show significant positive correlations (r = + 0.83) and (r = + 0.59) between the two factors in alkali and adjoining soil profiles respectively. Jackman (1955) reported similar results in a number of soils.

TABLE 1

Analytical data of alkali and adjoining soil profiles

Depth in inches	Calcium carbonate %	pH	Phosphorus Total %	Phosphorus Organic %	Phosphorus Inorganic %	Ratio Organic P / Total P
			1. Abhanpur (Ballia)-Alkali soil.			
0-2	0.2	9.4	0.02238	0.0064	0.01598	0.29
2-10	0.3	9.3	0.01827	0.0051	0.01317	0.28
10-20	0.7	9.2	0.01758	0.0044	0.01318	0.25
20-35	1.5	9.0	0.02299	0.0036	0.01939	0.16
35-50	1.4	8.9	0.02367	0.0048	0.01887	0.16
50-65	15.0	8.8	0.03110	0.0042	0.02690	0.10
65-74	22.0	8.8	0.01355	0.0036	0.00995	0.27
			2. Abhanpur (Ballia)-Adjoining soil.			
0-8	1.3	8.0	0.02494	0.0052	0.01974	0.21
8-20	16.6	8.3	0.03110	0.0058	0.02530	0.19
20-32	19.0	8.4	0.03381	0.0057	0.02811	0.17
32-48	25.2	8.6	0.02367	0.0055	0.01817	0.23
48-65	16.0	8.2	0.03178	0.0039	0.02788	0.12
65-72	26.5	8.1	0.01357	0.0041	0.00947	0.30

TABLE 1 – *Contd.*

Depth in inches	Calcium carbonate %	pH	Phosphorus Total %	Organic %	Inorganie %	Ratio Organic F Total P
			3. Lagdharpur (Jaunpur)-Alkali soil.			
0-2	0.2	9.3	0,01439	0.0070	0.00739	0.49
2-8	0.2	9.2	0.03110	0.0066	0.02450	0.21
8-20	0.8	9.0	0.04059	0.0065	0.03409	0.16
20-32	9.0	8.8	0.02807	0.0067	0.02137	0.24
32-44	7.1	8.9	0.03115	0.0044	0.02675	0.14
44-62	1.0	9.1	0.03313	0.0059	0.02713	0.18
			4. Lagdharpur (Jaunpur)-Adjoining soil.			
0-6	0.8	8.1	0.04063	0.0047	0.03593	0.12
6-19	Nil	7.7	0.04064	0.0070	0.03364	0.16
19-35	5.4	7.7	0.03381	0.0047	0.02911	0.14
35-52	7.9	7.8	0.02303	0.0046	0.01843	0.20
52-65	12.5	7.9	0.03043	0.0028	0.02763	0.09
			5. Korajpur (Varanasi)-Alkali soil.			
0-2	0.3	9.0	0.01269	0.0037	0.00899	0.29
2-12	Nil	8.6	0.00676	0.0017	0.00503	0.25
12-24	Nil	8.4	0.00540	0.0018	0.00360	0.33
24-36	Nil	8.1	0.00406	0.0012	0.00286	0.30
36-50	Nil	7.8	0.00338	0.0012	0.00218	0.36
50-66	0.4	7.7	0,00880	0.0020	0.00600	0.23
66-75	0.6	7.7	0.01149	0.0043	0.00719	0.37
			6. Korajpur (Varanasi)-Adjoining soil.			
0-10	Nil	7.2	0.01825	0.0011	0.01715	0.06
10-24	Nil	7.2	0.01827	0.0013	0.01796	0.07
24-36	Nil	7.3	0.02231	0.0041	0.01821	0.18
36-48	Nil	7.3	0.01434	0.0049	0.00944	0.34
48-60	Nil	7.4	0.01165	0.0022	0.00945	0.19
60-70	0.3	7.4	0.01759	0.0028	0.01479	0.16

Kanwar and Grewal (1959) reported an increase in organic and inorganic phosphorus contents with a decrease in soil pH, while Jackman (1955) found no significant relationship between pH and organic phosphorus in New Zealand soils. To study the relationships between pH and organic and inorganic forms of phosphorus, the data were analysed statistically which gave a significant positive correlation (r = + 0.71) in alkali and (r = + 0.59) in adjoining soil profiles between organic phosphorus and pH. The correlations for pH and inorganic phosphorus were found to be significant (r = + 0.54) in alkali soil profiles and not significant (r = + 0. 42) in adjoining soil profiles.

To study the relationship between $CaCO_3$ and organic and inorganic phosphorus, the data were analysed statistically. It shows no significant correlation (r = + 0.06) and (r = + 0.30) between $CaCO_3$ and organic phosphorus in alkali and adjoining soil profiles respectively. The correlations for $CaCO_3$ and inorganic phosphorus were found to be not significant (r= + 0.23) and (r= +0.02) in alkali and adjoining soil profiles respectively. Hoyos and Garcia (1959) reported dissimilar results in a number of soils.

Several investigators (Jackman 1955; Thompson *et al.*, 1954 and Youeda and Shigeta, 1956) have reported a positive relationship between organic phosphorus and organic carbon contents of the soils. A perusal of table 2 indicates that organic phosphorus content varies irrespective of the total organic carbon content of the soil, showing no apparent relationship between the two factors in both the types of soil profiles. Statistical analysis of the data also shows no significant correlation (r = + 0.04) and (r = + 0.08) in the alkali and adjoining soil profiles respectively. The result thus substantiates the findings of Kaila (1963).

Williams and Steinbergs (1958) reported a significant relationship between organic phosphorus and nitrogen level in the soil. Thompson *et al.* (1954) found a positive relationship between the two factors. The results reveal (Table 2) that organic phosphorus content does not seem to bear any apparent relationship with the total nitrogen content of the soil. Statistical analysis of the data for the two factors also shows no significant correlation (r = – 0.23) and (r = + 0.48) in alkali and adjoining soil profiles respectively. These results lead to indicate that probably a very small fraction of total nitrogen is associated with soil organic phosphorus.

TABLE 2

Organic Phosphorus affected by organic carbon, nitrogen and sulphur in the soil (Average in per cent.)

Organic carbon range %	Organic Phosphorus		Total Nitrogen range %	Organic Phosphorus		Total Sulphur range %	Organic Phosphorus	
	Alkali	Cultivated		Alkali	Cultivated		Alkali	Cultivated
0-0.1	...	...	0-0.030	0.0051	...	0-0.02	0.0023	0.0036
0.2-0.3	0.0047	...	0.030—0.035	0.0054	...	0.02—0.03	0.0057	0.0043
0.4-0.5	0.0060	0.0052	0.035—0.040	0.0040	0.0030	0.03—0.04	0.0054	0.0041
0.6-0.7	0.0042	0.0041	0.040—0.045	0.0035	0.0041	0.04—0.05	0.0052	0.0052
>0.7	0.0036	·0.0036	>0.045	0.0044	0.0044	>0.05	0.0047	0.0057

Further, an examination of table 2 does not show any apparent relationship between total sulphur and organic phosphorus in both the types of soil profiles. But statistical analysis of the data for the two factors reveals significant positive correlations (r = + 0.47) and (r = + 0.51) both at 5% level in alkali and adjoining soil profiles respectively. The statistical results evidently show a close relationship between organic phosphorus and total sulphur content of the soil (Williams *et al.*, 1960 and Walker & Adams, 1958).

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SOME SERIES OF MEIJER'S G-FUNCTION

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ABSTRACT

In this paper, the author has summed some series of Meijer's G-function by expressing the G-function as Mellion-Barnes type integral and then interchanging the order of integration and summation.

1. In this paper we have summed a number of series of Meijer's G-function, expressing the G-function as Mellin-Barnes type integral and then interchanging the order of integration and summation. MacRobert (7, 8, 9) summed many such series for E-function, which is itself a particular case of Meijer's G-function. Recently Bhise (1, 2) has summed many such series.

2. The Mellin-Barnes type integral (6, p. 207) which we have employed is

$$(2.1). \quad G^{l,\,u}_{p,\,q}\left(x \,\middle|\, \begin{matrix} a_1, \ldots\ldots\ldots\ldots, a_p \\ b_1, \ldots\ldots\ldots\ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_L \frac{\prod\limits_{j=1}^{l} \Gamma(b_j - s) \prod\limits_{j=1}^{u} \Gamma(1 - a_j + s)}{\prod\limits_{j=l+1}^{q} \Gamma(1 - b_j + s) \prod\limits_{j=u+1}^{p} \Gamma(a_j - s)} x^s \, ds$$

where an empty product is interpreted as 1, $0 \leqslant l \leqslant q$, $0 \leqslant u \leqslant p$ and the path L of integration runs from $-i\infty$ to $+i\infty$ so that all the poles of $\Gamma(b_j - s)$, $j = 1, 2, \ldots, l$ are to the right and all the poles of $\Gamma(1 - a_j + s)$, $j = 1, 2, \ldots\ldots, u$ to the left of L. The formula is valid for $p + q < 2(l + u)$ and $|\arg x| < (l + u - \frac{1}{2}p - \frac{1}{2}q)\pi$.

We have also used the following known formulae. In what follows λ, n and r are positive integers.

The multiplication formula for the Gamma function is

$$(2.2). \quad \Gamma(nz) = (2\pi)^{(1-n)/2} \, n^{nz - \frac{1}{2}} \prod_{i=0}^{n-1} \Gamma\left(z + \frac{i}{n}\right)$$

From (2.2), we also have

$$(2.3). \quad \prod_{i=0}^{n-1} \Gamma\left(\frac{a + r + i}{n}\right) = n^{-r} (a)_r \prod_{i=0}^{n-1} \Gamma\left(\frac{a + i}{n}\right)$$

$$(2.4).\quad \prod_{i=0}^{n-1} \Gamma\left(\frac{a+r-i}{n}\right) = n^{-r}(a-n+1)_r \prod_{i=0}^{n-1} \Gamma\left(\frac{a-i}{n}\right)$$

and

$$(2.5).\quad \prod_{i=0}^{n-1} \Gamma\left(\frac{a-r+i}{n}\right) = \frac{(-n)^r}{(1-a)_r} \prod_{i=0}^{n-1} \Gamma\left(\frac{a+i}{n}\right)$$

where $(a)_r = a(a+1)(a+2)\ldots\ldots(a+r-1)$. Also

$$(2.6).\quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)}$$

Modified form of Whipple's theorem as given by Dzrbasjan (5) is

$$(2.7).\quad {}_3F_2\,(a,\ 1-a,\ -n;\ f,\ 1-2n-f;\ 1)$$

$$= 2^{2n} \frac{(\tfrac{1}{2}a+\tfrac{1}{2}f)_n\,(\tfrac{1}{2}a-n-\tfrac{1}{2}f+\tfrac{1}{2})_n}{(f)_n\,(1-2n-f)_n}$$

Cartliz (4) has shown that

$$(2.8).\quad {}_4F_3\left[\begin{matrix} n,\ \tfrac{1}{2}(\alpha+1),\ \tfrac{1}{2}\alpha+1,\ \beta+n \\ \alpha+1,\ \tfrac{1}{2}(\beta+1),\ \tfrac{1}{2}\beta+1 \end{matrix};1\right] = \frac{\beta\,(\beta-\alpha)_n}{(\beta+2n)\,(\beta)_n}$$

Dougall's 1st theorem (10, p. 371) is

$$(2.9).\quad F\left[\begin{matrix} \alpha,\ 1+\tfrac{1}{2}\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ -n \\ \tfrac{1}{2}\alpha,\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha-\delta+1,\ \alpha-\epsilon+1,\ \alpha+n+1 \end{matrix}\right] =$$

$$= \frac{(\alpha+1)_n\,(\alpha-\beta-\gamma+1)_n\,(\alpha-\gamma-\delta+1)_n\,(\alpha-\delta-\beta+1)_n}{(\alpha-\beta+1)_n\,(\alpha-\gamma+1)_n\,(\alpha-\delta+1)_n\,(\alpha-\beta-\gamma-\delta+1)_n}$$

provided $1+2\alpha = \beta+\gamma+\delta+\epsilon-n$.

Dixion's theorem (10, p. 362) is

$$(2.10).\quad F\left[\begin{matrix} \alpha,\ \beta,\ \gamma \\ \alpha-\beta+1,\ \alpha-\gamma+1, \end{matrix};1\right] =$$

$$= \frac{\Gamma(\tfrac{1}{2}\alpha+1)\ \Gamma(\alpha-\beta+1)\ \Gamma(\alpha-\gamma+1)\ \Gamma(\tfrac{1}{2}\alpha-\beta-\gamma+1)}{\Gamma(\alpha+1)\ \Gamma(\tfrac{1}{2}\alpha-\beta+1)\ \Gamma(\tfrac{1}{2}\alpha-\gamma+1)\ \Gamma(\alpha-\beta-\gamma+1)}$$

where $R\,(\alpha-2\beta-2\gamma) > -2$.

Gauss's theorem (10, p. 144) is

$$(2.11).\quad F\,(a,\ b;\ c;\ 1) = \frac{\Gamma(c)\ \Gamma(c-a-b)}{\Gamma(c-a)\ \Gamma(c-b)}, \qquad R\,(c-a-b) > 0,$$

3. Firstly, if $p+q<2(l+u)$, then

$$(3.1).\ \sum_{r=0}^{n} \frac{{}^{n}C_r\,(2\lambda)^{2r}}{\Gamma(2\beta+r)\;\Gamma(1-2\beta-2n+r)}$$

$$\times\, G^{l+2\lambda,\,u}_{p+2\lambda,\,q+2\lambda}\left(x \,\middle|\, \begin{matrix} a_1,\ldots\ldots, a_p, \Delta(2\lambda, 2\alpha-r) \\ \Delta(2\lambda, 2\alpha+r), b_1\ldots\ldots b_q \end{matrix} \right)$$

$$= \frac{(2\lambda)^{2n}}{\Gamma(2\beta+n)\;\Gamma(1-2\beta-n)}$$

$$\times\, G^{l+2\lambda,\,u}_{p+2\lambda,\,q+2\lambda}\left(x \,\middle|\, \begin{matrix} a_1,\ldots\ldots, a_p, \Delta(\lambda, \alpha+\beta), \Delta(\lambda, \alpha-\beta-n+\tfrac{1}{2}) \\ \Delta(\lambda, \alpha+\beta+n), \Delta(\lambda, \alpha-\beta+\tfrac{1}{2}), b_1,\ldots\ldots b_q \end{matrix} \right)$$

where $\Delta(m, z)$ represents the set of parameters

$$\frac{z}{m}, \frac{z+1}{m}, \ldots\ldots\ldots\ldots, \frac{z+m-1}{m}$$

Substituting on the left from (2.1), we get

$$\sum_{r=0}^{n} \frac{{}^{n}C_r\,(2\lambda)^{2r}}{\Gamma(2\beta+r)\;\Gamma(1-2\beta-2n+r)}\,\frac{1}{2\pi i}$$

$$\times \int \frac{\prod_{j=0}^{2\lambda-1}\Gamma\left(\frac{2\alpha+r+j}{2\lambda}-s\right)\prod_{j=1}^{l}\Gamma(b_j-s)\prod_{j=1}^{u}\Gamma(1-a_j+s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\prod_{j=u+1}^{p}\Gamma(a_j-s)\prod_{j=0}^{2\lambda-1}\Gamma\left(\frac{2\alpha-r+j}{2\lambda}-s\right)}\,x^s ds\,.$$

Changing the order of summation and integration, applying (2.3) and (2.5), we have

$$\frac{1}{\Gamma(2\beta)\;\Gamma(1-2\beta-2n)}\;\frac{i}{2\pi i}\int \frac{\prod_{j=1}^{l}\Gamma(b_j-s)\prod_{j=1}^{u}\Gamma(1-a_j+s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\prod_{j=u+1}^{p}\Gamma(a_j-s)}\,x^s\, I\, ds,$$

where $\quad I = F\left[\begin{matrix}2\alpha-2\lambda s, & 1-2\alpha+2\lambda s, & -n \\ 2\beta, & 1-2\beta-2n & \end{matrix};1\right]$

Now, from (2.7) we have

$$I = 2^{2n}\frac{(\alpha+\beta-\lambda s)_n\,(a-\beta-n+\frac{1}{2}-\lambda s)_n}{(2\beta)_n\quad(1-2\beta-2n)_n}$$

Hence, applying (2.6), (2.2) and (2.1), the result is obtained.

Putting $\alpha = 1$, $l=q$. $a_1=1$, replacing a_i by a_{i-1}; $i=2, 3, \ldots\ldots, p$ and using [6, p. 215, 5.6 (2)], (3.1) reduces to

$$(3.2).\ \sum_{r=0}^{n}\frac{{}^nC_r\,(2\lambda)}{\Gamma(2\beta+r)\quad\Gamma(1-2\beta-2n+r)}E\left[\begin{matrix}\triangle(2\lambda,\ 2\alpha+r), b_1,\ldots\ldots\ldots\ldots, b_q\ ;x\\ \triangle(2\lambda,\ 2\alpha-r), a_1\ldots\ldots\ldots\ldots, a_{p-1}\end{matrix}\right]$$

$$=\frac{(2\lambda)^{2n}}{\Gamma(2\beta+n)\quad\Gamma(1-2\beta-n)}$$

$$\times\ E\left[\begin{matrix}\triangle(\lambda,\ \alpha+\beta+n),\ \triangle(\lambda,\ \alpha-\beta+\frac{1}{2})\ b_1,\ldots\ldots\ldots\ldots, b_q;\ x\\ \triangle(\lambda,\ a+\beta),\ \triangle(\lambda,\alpha-\beta-n+\frac{1}{2}), a_1,\ldots\ldots\ldots\ldots, a_{p-1}\end{matrix}\right]$$

Next, if $p+q<2\,(l+u)$, then

$$(3.3).\ \sum_{r=0}^{n}\frac{{}^nC_r\ \Gamma(\beta+n+r)\quad(-\lambda)^r}{\Gamma(1+\beta+2r)}G^{l+\lambda,\ u}_{p+\lambda,\ q+\lambda}\left(x\middle|\begin{matrix}a_1,\ldots\ldots\ldots, a_p,\ \triangle(\lambda,\ \alpha+r)\\ \triangle(\lambda,\ \alpha+2r),\ b_1,\ldots\ldots, b_q\end{matrix}\right)$$

$$=\frac{(-\lambda)^n}{(\beta+2n)}G^{l+\lambda,\ u}_{p+\lambda,\ q+\lambda}\left(x\middle|\begin{matrix}a_1\ldots\ldots\ldots, a_p,\ \triangle(\lambda,\ \alpha-\beta-n)\\ \triangle(\lambda,\ \alpha-\beta),\ b_1,\ldots\ldots\ldots, b_q\end{matrix}\right)$$

For the series is equal to

$$\sum_{r=0}^{n}\frac{{}^nC_r\ \Gamma(\beta+n+r)\quad(-\lambda)^r}{\Gamma(1+\beta+2r)}\ \frac{1}{2\pi i}\times$$

$$\times\int\frac{\prod\limits_{j=0}^{\lambda-1}\Gamma(\frac{\alpha+2r+j}{\lambda}-s)\ \prod\limits_{j=1}^{l}\Gamma(b_j+s)\ \prod\limits_{j=1}^{u}\Gamma(1-a_j+s)}{\prod\limits_{j=l+1}^{q}\Gamma(1-b_j+s)\ \prod\limits_{j=u+1}^{p}\Gamma(a_j-s)\ \prod\limits_{j=0}^{\lambda-1}\Gamma(\frac{\alpha+r+j}{\lambda}-s)}\ x^r\ ds.$$

Changing the order of summation and integration, using (2.3) and the relation

$(a)_{2n}=2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n$, we have the series equal to

$$\frac{\Gamma(\beta+n)}{\Gamma(1+\beta)}\frac{1}{2\pi i}\int\frac{\prod_{j=1}^{l}\Gamma(b_j-s)\prod_{j=1}^{u}\Gamma(1-a_j+s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\prod_{j=u+1}^{p}\Gamma(a_j-s)}x^s\,I\,ds,$$

where $I=F\left[{-n,\ \frac{1}{2}(\alpha-\lambda s),\ \frac{1}{2}(\alpha-\lambda s+1),\ \beta+n;\ 1 \atop \alpha-\lambda s,\ \frac{1}{2}(\beta+1),\ \frac{1}{2}\beta+1}\right]$

Using (2.8), (2.6), (2.2) and (2.1) the result is obtained.

Putting $u=1$, $l=q$, $a_1=1$, replacing a_i by a_{i-1}, $i=2, 3,\ldots\ldots$, p and using [6, p. 215, 5.6 (2)], (3.3) reduces to

$$(3.4).\ \sum_{r=0}^{n}\frac{{}^nC_r\,\Gamma(\beta+n+r)\,(-\lambda)^r}{\Gamma(1+\beta+2r)}E\left[{\Delta(\lambda,\ \alpha+2r),\ b_1,\ldots\ldots,b_q;\ x \atop \Delta(\lambda,\ \alpha+r),\ a_1,\ldots\ldots,a_{p-1}}\right]$$

$$=\frac{(-\lambda)^n}{\beta+2n}E\left[{\Delta(\lambda,\ \alpha-\beta),\ b_1,\ldots\ldots,b_q;\ x \atop \Delta(\lambda,\ \alpha-\beta-n),\ a_1,\ldots\ldots,a_{p-1}}\right]$$

Putting $u=0$, $\lambda=1$, $l=1$, $q=1$; using [6, p. 216, 5.6 (6)]; and replacing $(\alpha-b_1)/2$ by m (3.3) reduces to

$$(3.5).\ \sum_{r=0}^{n}\frac{{}^nC_r\,\Gamma(\beta+n+r)\ (-1)^{n+r}}{\Gamma(1+\beta+2r)}x^r\,W_{\frac{1}{2}-m,\ m+}(x)$$

$$=\frac{x^{-\frac{1}{2}\beta}}{\beta+2n}W_{\frac{1}{2}-m+\frac{1}{2}\beta+n,\ m-\frac{1}{2}\beta}(x)$$

Now we use Dougall's 1st theorom to prove the follwing summation. If $p+q<2\,(l+u)$ and $2\alpha+\gamma+\epsilon+n=\beta+\delta+1$, then

$$(3.6).\ \sum_{r=0}^{n}\frac{(-1)^r\,{}^nC_r\,(\alpha+2r)\ \Gamma(\alpha+r)}{\Gamma(\alpha+n+r+1)}\times$$

$$\times G^{l+4\lambda,\,u}_{p+4\lambda,\,q+4\lambda}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p,\ \triangle(\lambda,\gamma-r),\ \triangle(\lambda,\alpha+\gamma+r),\ \triangle(\lambda,\epsilon-r),\ \triangle(\lambda,\alpha+\epsilon+r)\\ \triangle(\lambda,\beta+r),\ \triangle(\lambda,\beta-\alpha-r),\ \triangle(\lambda,\delta+r),\ \triangle(\lambda,\delta-\alpha-r),\ b_1,\dots,b_q\end{matrix}\right)$$

$$=\left(\frac{2}{\lambda 3}\right)^n (\alpha-\beta+\gamma)_n(\alpha+\gamma-\delta)_n\ \times$$

$$\times G^{l+6\lambda,\,u}_{p+6\lambda,\,q+6\lambda}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p,\ \triangle(\lambda,\gamma),\ \triangle(\lambda,\alpha+\gamma+n),\ \triangle(\lambda,\epsilon),\ \triangle(\lambda,\alpha+\epsilon+n),\ \triangle(2\lambda,\beta+\delta-\alpha-n)\\ \triangle(\lambda,\beta),\ \triangle(\lambda,\beta-\alpha-n),\ \triangle(\lambda,\delta),\ \triangle(\lambda,\delta-\alpha-n),\ \triangle(2\lambda,\beta+\delta-\alpha),\ b_1,\dots,b_q\end{matrix}\right)$$

Proceeding as before and noting that $\alpha+2r=\alpha\,(\frac{1}{2}\alpha+1)_r/(\frac{1}{2}\alpha)_r$, using (2.3) and (2.5) we get the series equal to

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}\ \frac{1}{2\pi i}\int\frac{\prod_{j=1}^{l}\Gamma(b_j-s)\prod_{j=1}^{u}\Gamma(1-a_j+s)\prod_{j=0}^{\lambda-1}\Gamma(\frac{\beta+j}{1}-s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\prod_{j=u+1}^{p}\Gamma(a_j-s)\prod_{j=0}^{\lambda-1}\left(\Gamma(\frac{\gamma+j}{\lambda}-s\right)}\ \times$$

$$\times\frac{\prod_{j=0}^{\lambda-1}\Gamma(\frac{\beta-\alpha+j}{\lambda}-s)\prod_{j=0}^{\lambda-1}\Gamma(\frac{\delta+j}{\lambda}-s)\prod_{j=0}^{\lambda-1}\Gamma(\frac{\delta-\alpha+j}{\lambda}-s)}{\prod_{j=0}^{\lambda-1}\Gamma(\frac{\alpha+\gamma+j}{\lambda}-s)\prod_{j=0}^{\lambda-1}\Gamma(\frac{\epsilon+j}{\lambda}-s)\prod_{i=0}^{\lambda-1}\Gamma(\frac{\alpha+\epsilon+j}{\lambda}-s)}\ x^s\,\mathrm{I}\,ds,$$

where $\mathrm{I}=F\left[\begin{matrix}\alpha,\ 1+\frac{1}{2}\alpha,\ \beta-\lambda s,\ 1-\gamma+\lambda s,\ \delta-\lambda s,\ 1-\epsilon+\lambda s,\ -n\\ \frac{1}{2}\alpha,\ \alpha-\beta+\lambda s+1,\ \alpha+\gamma-\lambda s,\ \alpha-\delta+\lambda s+1,\ \alpha+\epsilon-\lambda s,\ \alpha+n+1\end{matrix};1\right]$

Using (2.9) and (2.6), we have

$$\mathrm{I}=(\alpha+1)_n(\alpha-\beta+\gamma)_n(\alpha+\gamma-\delta)_n$$

$$\times\frac{\Gamma(\beta+\delta-\alpha-2\lambda s)\quad\Gamma(\beta-\alpha-n-\lambda s)}{\Gamma(\beta+\delta-\alpha-n-2\lambda s)\quad\Gamma(\beta-\alpha-\lambda s)}\ \times$$

$$\times \frac{\Gamma(\alpha+\gamma-\lambda s) \quad \Gamma(\delta-\alpha-n-\lambda s) \quad \Gamma(\beta+\delta-\alpha-\gamma-n+1-\lambda s)}{\Gamma(\alpha+\gamma+n-\lambda s) \quad \Gamma(\delta-\alpha-\lambda s) \quad \Gamma(\beta+\delta-\alpha-\gamma+1-\lambda s)}.$$

Hence using (2.2) and (2.1) the result follows.

Putting $u=1$, $l=q$, $a_1=1$; replacing a_i by a_{i-1}, $i=2, 3, \ldots \ldots p$ and using [6, p. 215, 5.6 (2)], (3.6) gives

$$(3.7). \quad \sum_{r=0}^{n} \frac{(-1)^r \, {}^nC_r \quad (\alpha+2r) \quad \Gamma(\alpha+r)}{\Gamma(\alpha+n+r+1)} \times$$

$$\times \; E\left[\begin{matrix}\Delta(\lambda, \beta+r), \Delta(\lambda, \beta-\alpha-r), \Delta(\lambda, \delta+r), \Delta(\lambda, \delta-\alpha-r), b_1, \ldots, b_q ; x \\ \Delta(\lambda, \gamma-r), \Delta(\lambda, \alpha+\gamma+r), \Delta(\lambda, \epsilon-r), \Delta(\lambda, \alpha+\epsilon+r), a_1, \ldots, a_{p-1}\end{matrix}\right]$$

$$=\left(\frac{2}{\lambda^3}\right)^n (\alpha-\beta+\delta)_n (\alpha+\gamma-\delta)_n \times$$

$$\times \; E\left[\begin{matrix}\Delta(\lambda, \beta), \Delta(\lambda, \beta-\alpha-n), \Delta(\lambda, \delta), \Delta(\lambda, \delta-\alpha-n), \\ \Delta(\lambda, \gamma), \Delta(\lambda, \alpha+\gamma+n), \Delta(\lambda, \epsilon), \Delta(\lambda, \alpha+\epsilon+n),\end{matrix}\right.$$

$$\left.\begin{matrix}\Delta(2\lambda, \beta+\delta-\alpha), \quad b_1, \ldots, b_q ; x \\ \Delta(2\lambda, \beta+\delta-\alpha-n), a_1, \ldots, a_{p-1}\end{matrix}\right]$$

provided $2\alpha+\gamma+\epsilon+n=\beta+\delta+1$.

Next, if $p+q<2(l+u)$, $|\arg x| < (l+u-\frac{1}{2}p-\frac{1}{2}q)\pi$ and $R(2\alpha-2\beta-3k)>0$, then

$$(3.8). \quad \sum_{r=0}^{\infty} \frac{(k)_r}{L(r)} G^{l+\lambda, u+\lambda}_{p+2\lambda, q+2\lambda}\left(x \,\middle|\, \begin{matrix}\Delta(\lambda, \alpha-k-r), a_1, \ldots, a_p, \Delta(\lambda, \alpha+r) \\ \Delta(\lambda, \beta+k+r), b_1, \ldots, b_q, \Delta(\lambda, \beta-r)\end{matrix}\right)$$

$$=\lambda^k \frac{\Gamma(\frac{1}{2}k+1) \quad \Gamma(\alpha-\beta-3/2\, k)}{\Gamma(k+1) \quad \Gamma(\alpha-\beta-k)}$$

$$G^{l+\lambda, u+\lambda}_{p+2\lambda, q+2\lambda}\left(x \,\middle|\, \begin{matrix}\Delta(\lambda, \alpha-k), a_1, \ldots, a_p, \Delta(\lambda, \alpha-\frac{1}{2}k) \\ \Delta(\lambda, \beta+k), b_1, \ldots, b_q, \Delta(\lambda, \beta+\frac{1}{2}k)\end{matrix}\right)$$

Proceeding as before, using (2.3) and (2.4), we have the series equal to

$$\frac{1}{2\pi i} \int \frac{\prod_{j=1}^{l} \Gamma(b_j-s) \prod_{j=1}^{u} \Gamma(1-a_j+s) \prod_{j=0}^{\lambda-1} \Gamma(\frac{\beta+k+j}{\lambda}-s) \prod_{j=0}^{\lambda-1} \Gamma(1-\frac{\alpha-k+j}{\lambda}+s)}{\prod_{j=l+1}^{q} \Gamma(1-b_j+s) \prod_{j=u+1}^{p} \Gamma(a_j-s) \prod_{j=0}^{\lambda-1} \Gamma(1-\frac{\beta+j}{\lambda}+s) \prod_{j=0}^{\lambda-1} \Gamma(\frac{\alpha+j}{\lambda}-s)} \times$$

$$\times x^s \, \mathrm{I}\, ds,$$

where $\mathrm{I}=F\left[\begin{matrix} k, 1-\alpha+k+\lambda s, \beta+k-\lambda s, \\ \alpha-\lambda s, \quad 1-\beta+\lambda s \end{matrix} ; 1\right]$

From (2.10), as have

$$\mathrm{I} = \frac{\Gamma(\tfrac{1}{2}k+1)\ \Gamma(\alpha-\beta-3/2\,k)\ \Gamma(\alpha-\lambda s)\quad \Gamma(1-\beta+\lambda s)}{\Gamma(k+1)\ \ \Gamma(\alpha-\beta-k)\qquad \Gamma(\alpha-\tfrac{1}{2}k-\lambda s)\ \ \Gamma(1-\beta-\tfrac{1}{2}k+\lambda s)}$$

Hence applying (2.2) and (2.1), the result is obtained.

Now, we establish few series by using Gauss's theorem.

If $\lambda \leqslant u \leqslant p$, $\lambda \leqslant l+\lambda \leqslant q$, $p+q < 2\,(l+u)$ and R $(\beta-\alpha) < n$, then

$$(3.9).\quad \sum_{r=0}^{n} (-1)^{n+r}\ {}^{n}C_r\ G_{p,q}^{l,u}\left(x \middle| \begin{matrix} \Delta(\lambda, \alpha-r), a_{\lambda+1} \ldots\ldots\ldots, a_p \\ b_1, \ldots\ldots\ldots, b_{q-\lambda}, \Delta(\lambda, \beta-r) \end{matrix} \right)$$

$$= \frac{\Gamma(1-\alpha+\beta)}{\lambda^n\ \Gamma(1-\alpha+\beta-n)}\, G_{p,q}^{l,u}\left(x \middle| \begin{matrix} \Delta(\lambda, \alpha), a_{\lambda+1} \ldots\ldots\ldots, a_p \\ b_1, \ldots\ldots, bq-\lambda, \Delta(\lambda, \beta-n) \end{matrix} \right)$$

From (2.1) and using (2.4), we have the series equal to

$$(-1)^n \frac{1}{2\pi i}\int \frac{\prod_{j=1}^{l}\Gamma(b_j-s)\ \prod_{j=\lambda+1}^{u}\Gamma(1-a_j+s)\ \prod_{j=0}^{\lambda-1}\Gamma\left(1-\frac{\alpha+j}{\lambda}+s\right)}{\prod_{j=l+1}^{q-\lambda}\Gamma(1-b_j+s)\ \prod_{j=u+1}^{p}\Gamma(a_j-s)\ \prod_{j=0}^{\lambda-1}\left(1-\frac{\beta+j}{\lambda}+s\right)}\, x^s\, \mathrm{I}\, ds,$$

where $\mathrm{I}=F\,(-n, 1-\alpha+\lambda s; 1-\beta+\lambda s; 1)$.

Using (2.11), (2.2) and (2.1), we get the result.

With $\lambda=1$, we get a known series [2, p. 271 (3.5)], and with $\lambda=1$, $n=1$, and β replaced by $\beta+1$, we obtain a known result [3, p. 353 (5.1) and (5.4)].

Similarly, if $\lambda \leqslant l \leqslant q$, $p+q < 2\,(l+u)$ and R $(a_p) < n+1$, then

$$(3.10.\quad \sum_{r=0}^{n} \frac{(-1)^{n+r}\ {}^{n}C_r\ \lambda^r}{\Gamma(1+b-a_p+r)}\, G_{p,q}^{l,u}\left(x \middle| \begin{matrix} a_1, \ldots\ldots\ldots\ldots\ldots, a_p \\ \Delta(\lambda, b+r), b_{\lambda+1}, \ldots\ldots\ldots\, bq \end{matrix} \right)$$

$$= \frac{\lambda^{u}}{\Gamma(1+b-a_p+n)} \; G^{l+\lambda,\, u}_{p+\lambda,\, q+\lambda}\left(x \,\middle|\, \begin{matrix} a_1 \ldots\ldots, a_p, \Delta(\lambda, a_p - n) \\ \Delta(\lambda, a_p), \Delta(\lambda, b), b_{\lambda+1}, \ldots\ldots, b_q \end{matrix} \right)$$

With $\lambda = 1$ and b replaced by b_1, we have a known series [1, p. 14 (4.1)].

Also, if $\lambda \leqslant u \leqslant p-\lambda$, $p+q < 2(l+u)$ and $R(\alpha)>0$, then

$$(3.11).\quad \sum_{r=0}^{n} (-1)^{n+r} \lambda^{-r}\, {}^{n}C_r\, \Gamma(\beta-\alpha+n+r) \times$$

$$\times G^{l,\, u}_{p,\, q}\left(x \,\middle|\, \begin{matrix} \Delta(\lambda, \alpha)\, a_{\lambda+1}, \ldots\ldots, a_{p-\lambda}, \Delta(\lambda, \beta+r) \\ b_1, \ldots\ldots\ldots, b_q \end{matrix} \right)$$

$$= \Gamma(\beta-\alpha+n)\; G^{l,\, u}_{p,\, q}\left(x \,\middle|\, \begin{matrix} \Delta(\lambda, \alpha-n), a_{\lambda+1}, \ldots\ldots, a_{p-\lambda}, \Delta(\lambda, \beta+n) \\ b_1, \ldots\ldots\ldots, b_q \end{matrix} \right)$$

Proceeding as usual and applying (2.3), (2.11), (2.6), (2.2) and (2.1), we get the result.

With $n=1$, $\lambda=1$ and replacing α by a_1 and $\beta+1$ by a_p; (3.11) yields

$$(a_p - a_1)\; G^{l,\, u}_{p,\, q}\left(x \,\middle|\, \begin{matrix} a_1, \ldots\ldots, a_p \\ b_1, \ldots\ldots, b_q \end{matrix} \right) = G^{l,\, u}_{p,\, q}\left(x \,\middle|\, \begin{matrix} a_1 - 1, a_2, \ldots\ldots, a_p \\ b_1, \ldots\ldots\ldots, b_q \end{matrix} \right)$$

$$+ G^{l,\, u}_{p,\, q}\left(x \,\middle|\, \begin{matrix} a_1, \ldots\ldots, a_{p-1}, a_p - 1 \\ b_1, \ldots\ldots, b_q \end{matrix} \right)$$

which is a known relation [5, p. 210 (12)].

Lastly, if $\lambda \leqslant 1 \leqslant q-\lambda$, $p-q < 2(l+u)$ and $R(\alpha) < 1$, then

$$(3.12).\quad \sum_{r=0}^{n} (-1)^{n+r} \lambda^{-r}\, {}^{n}C_r\, \Gamma(\alpha-\beta+n+r) \times$$

$$\times G^{l,\, u}_{p,\, q}\left(x \,\middle|\, \begin{matrix} a_1, \ldots\ldots\ldots, a_p \\ \Delta(\lambda, \alpha), b_{\lambda+1}, \ldots\ldots, b_{q-\lambda}, \Delta(\lambda, \beta-r) \end{matrix} \right)$$

$$= \Gamma(\alpha-\beta+n)\, G^{l,\, u}_{p,\, q}\left(x \,\middle|\, \begin{matrix} a_1, \ldots\ldots\ldots, a_p \\ \Delta(\lambda, \alpha+n), b_{\lambda+1}, \ldots\ldots, b_{q-\lambda}, \Delta(\lambda, \beta-n) \end{matrix} \right)$$

With $n=1$, $\lambda=1$ and replacing α by b_1 and $\beta-1$ by b_q, (3.12) yields

$$(b_1-b_q)\, G^{l,u}_{p,q}\left(x\,\Big|\,\begin{matrix}a_1, \ldots\ldots\ldots\ldots, a_p\\ b_1 \ldots\ldots\ldots\ldots; b_q\end{matrix}\right)$$

$$= G^{l,u}_{p,q}\left(x\,\Big|\,\begin{matrix}a_1, \ldots\ldots\ldots\ldots\ldots\ldots, a_p\\ b_1+1, b_2, \ldots\ldots\ldots\ldots, b_q\end{matrix}\right)$$

$$+ G^{l,u}_{p,q}\left(x\,\Big|\,\begin{matrix}a_1, \ldots\ldots\ldots\ldots\ldots\ldots, a_p\\ b_1, \ldots\ldots\ldots\ldots, b_{q-1}, b_q+1\end{matrix}\right)$$

which is a known result [3, p. 353, (5.3) and (5.4)].

I wish to express my sincere thanks to Dr. V. M. Bhise for his help and guidance in the preparation of this paper. My thanks are also due to the Principal Mr. V. V. Sarwate for the facilities he gave to me.

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A NOTE ON G-FUNCTIONS

By

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ABSTRACT

The present paper aims to evaluate two definite integrals involving the product of Meijer's G-function with Gauss's hypergeometric functions. These are analogous to the results recently given by Sharma ((3), p. 539).

1. The first result to be established is

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\,{}_2F_1\left(\begin{matrix}a,\ b\\ \tfrac12 a+\tfrac12 b+\tfrac12\end{matrix};x\right) G^{k\,l}_{pq}\left(zx^m(1-x)^m\middle|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)dx$$

$$=\frac{\pi}{2^{2\rho-1}m^{1/2}}\,\frac{\Gamma(\tfrac12+\tfrac12 a+\tfrac12 b)}{\Gamma(\tfrac12+\tfrac12 a)\,\Gamma(\tfrac12+\tfrac12 b)}\,G^{k,\,l+2m}_{p+2m,\,q+2m}\left(\frac{z}{4^m}\middle|\right.$$

$$\left.\begin{matrix}\frac{1-\rho}{m},\ \frac{2-\rho}{m},\ \dots,\frac{m+\rho}{m},\ \frac{1-2\rho+a+b}{2m},\ \frac{3-2\rho+a+b}{2m},\ \dots,\frac{2m-2\rho-1+a+b}{2m},\ a_1,\dots a_p\\ b_1,\dots,b_q,\ \frac{1-2\rho+a}{2m},\ \frac{3-2\rho+a}{2m},\ \dots,\frac{2m-1-2\rho+a}{2m},\ \frac{1-2\rho+b}{2m},\ \frac{3-2\rho+b}{2m},\ \dots,\frac{2m-1-2\rho+b}{2m}\end{matrix}\right), \quad (1)$$

where m is a positive integer, $R(\rho+mb_h)>0$, $h=1,\dots,k$, $R(a+b)>-1$, $2(k+l)>p+q$ and $|\arg z|<(k+l-\tfrac12 p-\tfrac12 q)\pi$.

The second result to be established is

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-b}\,{}_2F_1\left(\begin{matrix}a,\ 1-a\\ b\end{matrix};x\right) G^{kl}_{pq}\left(zx^m(1-x)^m\middle|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)dx$$

$$=\frac{\pi\,\Gamma(b)}{m^{1/2}\,\Gamma(\tfrac12 a+\tfrac12 b)}\,\frac{2^{1-2\rho}}{\Gamma(\tfrac12-\tfrac12 a+\tfrac12 b)}$$

$$\times\, G^{k,\; l+2m}_{p\;\; 2m,\; q+2m}\left(\frac{z}{4^m}\;\middle|\;\begin{matrix}\frac{1-\rho}{m},\ \frac{2-\rho}{m},\ \ldots,\frac{m-\rho}{m},\ \frac{b-\rho}{m},\ \frac{1+b-\rho}{m},\ldots,\ \frac{m-1+b-\rho}{m},\\ b_1,\ldots,b_q,\ \frac{1+b-a-2\rho}{2m},\ \frac{3+b-a-2\rho}{2m},\ \ldots,\frac{2m-1+b-a-2\rho}{2m},\end{matrix}\right.$$

$$a_1,\ \ldots,\ a_p$$

$$\left.\frac{a+b-2\rho}{2m},\ \frac{2+a+b-2\rho}{2m},\ \ldots\frac{2m-2+a+b-2\rho}{2m}\right), \tag{2}$$

where m is a positive integer, $R\,(\rho+m\,b_h)>0$, $R\,(\rho-b+m\,b_h)>0$, $h=1,\ldots,k$; $2\,(k+l)>p+q$ and $|\arg z|<(k+l-\frac{1}{2}p-\frac{1}{2}q)\,\pi$.

For establishing the above results we require the following formulae ((1), pages 207: 4, 189, (2) pages 104) :

$$G^{m\,n}_{p\,q}\left(z\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,z^s\,ds. \tag{3}$$

$$\Gamma(mz)=m^{mz-\frac{1}{2}}\,(2\pi)^{\frac{1}{2}-\frac{1}{2}m}\prod_{r=0}^{m-1}\Gamma\left(z+\frac{r}{m}\right) \tag{4}$$

$${}_3F_2\left[\begin{matrix}a,\ b,\ c;\\ \frac{1}{2}(a+b+1),\ 2c\end{matrix}\right]=\frac{\sqrt{\pi}\,\Gamma(c+\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\,\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})\,\Gamma(c+\frac{1}{2}-\frac{1}{2}a)\,\Gamma(c+\frac{1}{2}-\frac{1}{2}b)},$$

$$R\,(c)>-\tfrac{1}{2},\ R\,(a+b)>-1,\ R(2c-a-b)>-1. \tag{5}$$

$${}_3F_2\left[\begin{matrix}a,\ 1-a,\ c\ ;\\ f,\ 2c+1-f\end{matrix}\right]=\frac{\pi\,\Gamma(f)\,\Gamma(2c+1-f)\quad 2^{1-2c}}{\Gamma(c+\frac{1}{2}a+\frac{1}{2}-\frac{1}{2}f)\,\Gamma(\frac{1}{2}a+\frac{1}{2}f)\,\Gamma(c+1-\frac{1}{2}a-\frac{1}{2}f)}\times$$

$$\times\frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}f)},$$

$$R\,(f)>0 \text{ and } R\,(2c-f)>-1, \tag{6}$$

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,{}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ \ \ ,b_q\end{matrix};cx\right]dx$$

$$= B(\alpha,\beta)\,{}_{p+1}F_{q+1}\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};c\right], \tag{7}$$

where $R(\alpha) > 0$, $R(\beta) > 0$ and the resultant series is convergent.

PROOF OF THE FIRST RESULT: If we substitute the value of the G-function from (3) in the integrand of (1), change the order of integration, evaluate the inner integral with the help of (7) and (5), then the value of the integral thus obtained is

$$\frac{\pi\,\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{2^{2\rho-1}\,\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(\frac{1}{2}+\frac{1}{2}b)}\times$$

$$\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{k}\Gamma(b_j-s)\ \prod_{j=1}^{l}\Gamma(1-a_j+s)}{\prod_{j=k+1}^{q}\Gamma(1-b_j+s)\ \prod_{j=l+1}^{p}\Gamma(a_j-s)}\times$$

$$\times\frac{\Gamma(\rho+ms)\,\Gamma(\rho+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+ms)}{\Gamma(\rho+\frac{1}{2}-\frac{1}{2}a+ms)\quad\Gamma(\rho+\frac{1}{2}-\frac{1}{2}b+ms)}\left(\frac{z}{4^m}\right)^s ds, \tag{8}$$

to which if we apply (4) and then interpret with the help of (3), we get (1).

The change in the order of integration is admissible for the conditions given in (1), as with them the integral (3) is absolutely convergent, the inner integral after the change of order of integration is absolutely convergent and the resulting integral (7) is also convergent.

The proof to the second integral is exactly on the same lines as given above except that the result (6) is used instead of (5).

My best thanks are due to Professor Brij Mohan for his kind suggestions.

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CERTAIN PROPERTIES OF WHITTAKER TRANSFORM

By

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ABSTRACT

In this paper two theorems on Whittaker transform have been established, by utilizing the integral representations of the Whittaker function $W_{k,m}(z)$.

1. Varma (3, p. 209 & 4, p. 17) gave the generalisations of the classical Laplace transform

$$(1.1)\quad \phi(s) = s\int_0^\infty e^{-st} f(t)\,dt, \quad R(s) > 0$$

In the forms

$$(1.2)\quad \phi(s) = s\int_0^\infty (st)^{m-\frac{1}{2}}\, e^{-\frac{1}{2}st}\, W_{k,m}(st)\, f(t)\,dt$$

and

$$(1.3)\quad \phi(s) = s\int_0^\infty (2st)^{-\frac{1}{4}}\, W_{k,m}(2st)\, f(t)\,dt$$

which are known as Varma and Whittaker transforms respectively.

With $k+m=\frac{1}{2}$, (1.2) reduces to (1.1) and with $k=\frac{1}{4}$, $m=\pm\frac{1}{4}$, (1.3) also reduces to (1.1). We shall represent (1.1), (1.2) and (1.3) symbolically by

$$\phi(s) \doteq f(t), \qquad \phi(s; k, m) = W[f(t); k, m]$$

and $\phi(s; k, m) \;\frac{k}{m}\; f(t)$ respectively.

In this paper we have established two theorems for the Whittaker transform. The results have been obtained by utilizing the integral representations of the Whittaker function $W_{k,m}(z)$.

2. THEOREM 1. If

$$\phi(s; k, m) \;\frac{k}{m}\; f(t)$$

and

$$\psi\,(s;\,k,\,m;\,n;\,y)\ \overset{k}{\underset{m}{-}}\ t^{\,n}\ \ e^{yt}\ f(t)$$

then

$$(2.1)\quad \phi\,(s;\,k,\,m) = \frac{2^{\lambda-k}\ s^{\,k-\lambda-\mu+5/4}}{\Gamma(2\lambda-2k)} \int_0^\infty y^{2\lambda-2k-1}\ (y+s)^{\mu-5/4}\ \times$$

$$\times\ {}_2F_1\left[{\lambda-k+\mu\pm m;\ -y/s \atop 2\lambda-2k}\right] \psi\,(y+s;\ \lambda,\ \mu;\ \lambda-k;-y)\ dy\ ;$$

provided R $(s) \geqslant s_0 > 0$, R $(\lambda-k) > 0$, the Whittaker transform of $\left| t^{\,\lambda-k}\ e^{-yt}\ f(t)\right.$ exists and the integral in (2.1) is absolutely convergent.

Proof :—We have (1, p. 440).

$$W_{k,\,m}\ (z) = \frac{z^{\lambda-k}\ e^{\frac{1}{2}z}}{\Gamma(\lambda-2k)} \int_1^\infty e^{-\frac{1}{2}zv} W_{\lambda,\,\mu}\ (zv)$$

$${}_2F_1\left[{\lambda-k+\mu\pm m\ ;1-v \atop 2\lambda-2k}\right] \times$$

$$\times\ (v-1)^{2\lambda-2k-1}\ v^{\mu-\frac{1}{2}}\ dv,$$

$z\neq 0$, $|\arg z| < \frac{1}{2}\pi$ and R $(\lambda-k) > 0$.

Replacing z by $2st$ and v by $1+y/s$, sabstituting for $W_{k,\,m}(2st)$ in

$$\phi\,(s;\,k,\,m) = s \int_0^\infty (2st)^{-\frac{1}{4}}\ W_{k,\,m}(2st)\ \ f(t)\ dt$$

and changing the order of integration, we have

$$\phi\,(s;\,k,\,m) = \frac{2^{\lambda-k}\ s^{k-\lambda-\mu+5/4}}{\Gamma(2\lambda-2k)} \int_0^\infty y^{2\lambda-2k-1}\ (y+s)^{\mu-\frac{1}{4}}\ \times$$

$$\times\ {}_2F_1\left[{\lambda-k+\mu\pm m\ ;-y/s \atop 2\lambda-2k}\right] \int_0^\infty (2t)^{-\frac{1}{4}}\ W_{\lambda,\,\mu}\ (2yt+2st)\ \ t^{\,\lambda-k}\ e^{-yt} f\,(t) dt\ dy;$$

from which (2.1) follows immediately.

Regarding the change of order of integration, we see that the y-integral is absolutely and uniformly convergent if $R(s) \geqslant s_0 > 0$ and $R(\lambda - k) > 0$, the t-integral is absolutely and uniformly convergent if the Whittaker transform of $\left| t^{\lambda-k} e^{-yt} f(t) \right|$ exists and the absolute convergence of the repeated integral has been assumed. Hence, the change of order of integration is justified by de la Vallee Poussin's theorem.

2.1. COROLLARY.

On putting $\lambda = \frac{1}{4}$, $\mu = \pm\frac{1}{4}$, the theorem reduces to :

If $\phi(s; k, m) \frac{k}{m} f(t)$ and $\psi(s; n) \doteqdot t^n f(t)$

then

$$(2.2) \quad \phi(s; k, m) = \frac{2^{\frac{1}{4}-k} s^{k+\frac{3}{4}}}{\Gamma(\frac{1}{2}-2k)} \int_0^\infty y^{-2k-\frac{1}{2}} (2y+s)^{-1} \times$$

$$\times {}_2F_1(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; \tfrac{1}{2}-2k; -y/s) \psi(2y+s; \tfrac{1}{4}-k)\, dy$$

provided $R(s) \geqslant s_0 > 0$, $R(\frac{1}{4}-k) > 0$, the Laplace transform of $\left| t^{\frac{1}{4}-k} f(t) \right|$ exists and the integral in (2.2) is absolutely convergent.

3. THEOREM 2. If

$$\phi(s; k, m) \frac{k}{m} f(t)$$

and

$$\varphi(s; k, m; n; a) = W\left[t^n e^{a/t} f(1/t) \,;\, k, m \right]$$

then

$$(3.1)\ \phi(s; k, m) = s\,(2s)^{\frac{1}{4}} \int_0^\infty x^{-\beta-\mu-1} \times$$

$$\times\, G^{2,\,1}_{2,\,4}\left(2sx \,\middle|\, \begin{matrix} \beta+\lambda-\frac{1}{4}, \frac{3}{4}-k \\ m, -m, \beta+\mu, \beta-\mu \end{matrix} \right) \varphi(x; \lambda, \mu; -\beta-\mu-5/4; a)\, dx$$

provided $R(s) \geqslant s_0 > 0$, $R(1-\beta\pm m\pm\mu) > 0$, the Varma

transform of $|\, t^{-\beta-\mu-5/4}\, e^{s/t} f(1/t)\,|$ exists and the integral in (3.1) is absolutely convergent.

Proof :—Meijer (2, p. 188) has shown that

$$W_{k,\, m}(z) = z^{\frac{1}{2}}\, e^{\frac{1}{2}z} \int_0^\infty v^{-\beta-\frac{1}{2}}\, e^{-\frac{1}{2}v} W_{\lambda,\, \mu}(v) \times$$

$$\times\, G^{2,\, 1}_{2,\, 4}\left(zv \,\middle|\, \begin{matrix} \beta+\lambda-\frac{1}{2},\ \frac{1}{2}-k \\ m,\ -m,\ \beta+\mu,\ \beta-\mu \end{matrix}\right) dv,$$

$R(1-\beta\pm m\pm\mu) > 0$.

Replacing z by $2st$ and v by x/t in this and substituting for $W_{k,\, m}(2st)$ in

$$\phi(s;\, k,\, m) = s\int_0^\infty (2st)^{-\frac{1}{4}}\, W_{k,\, m}(2st)\, f(t)\, dt$$

and changing the order of integration, we have

$$\phi(s;\, k,\, m) = s\,(2s)^{\frac{1}{4}} \int_0^\infty x^{-\beta-\frac{1}{2}}\quad G^{2,\, 1}_{2,\, 4}\left(2sx \,\middle|\, \begin{matrix} \beta+\lambda-\frac{1}{2},\ \frac{1}{2}-k \\ m,\ -m,\ \beta+\mu,\ \beta-\mu \end{matrix}\right) \times$$

$$\times \int_0^\infty t^{\beta-\frac{1}{4}}\quad e^{st-x/(2t)}\, w_{\lambda,\, \mu}(x/t)\, f(t)\, dt\, dx.$$

This gives the result on putting $t = 1/y$.

The change of order of integration can be seen to be justified as in Theorem 1, under the conditions stated above.

3.1. COROLLARY.

On putting $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$, $\lambda + \mu = \frac{1}{2}$ and replacing $\beta + \lambda + \frac{1}{4}$ by $-\nu$ we get :

If

$$\phi(s) \doteqdot f(t) \qquad \text{and } \psi(s;\, n;\, a) \doteqdot t^{n}\, e^{a/t}\, f(1/t)$$

then

$$(3.2) \quad \Phi(s) = s(2s)^{-\frac{1}{2}\nu} \int_0^\infty x^{\frac{1}{2}\nu-1} J_\nu(2\sqrt{2sx})\,\psi(x, \nu-1; s)\,dx$$

provided $R(s) \geqslant s_0 \geqslant 0$, $R(1+\nu) > 0$, the Laplace

transform of $| t^{\nu-1} e^{s/t} f(1/t) |$ exists and the integral in (3.2) is is absolutely eonvergent.

I wish to express my sincerc thanks to Dr. V. M. Bhise for his help and guidance in the preparation of this paper. My thanks are also due to Principal V. V. Sarwate for the facilities he gave to me.

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THE H-FUNCTION—II

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ABSTRACT

In this paper first we give some of the important properties and special cases of the H-function. Later on we evaluate an infinite integral involving product of two H-functions. Since the H-function is a very general function, several integrals involving the product of the H-function with Wright's generalized hypergeometric function, Macrobert's E-function, Whittaker fnnction modified Bessel functton etc., follow as special cases of the integral evaluted earlier.

1. *The H-function.* The H-function introduced by Fox [5, p. 408], will be represented and defined as follows [6].

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_b) \\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_q) \end{matrix}\right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j \xi) \prod_{j=n+1}^{p} \Gamma(a - a_j \xi)} x^{\xi} \, d\xi \tag{1.1}$$

where x is not equal to zero and an empty product is interpreted as unity; p, q, m, n are integers satisfying $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; α_j $(j = 1, \ldots\ldots, p)$, β_j $(j = 1, \ldots\ldots q)$, are positive numbers and a_j $(j = 1, \ldots\ldots, \mathrm{p})$, b_j $(j = 1, \ldots\ldots, q)$, are complex numbers such that no pole of $\Gamma(b_h - \beta_h \xi)$ $(h = 1, \ldots\ldots, m)$, coincides with any pole of $\Gamma(1 - ai + \alpha_i \xi)$ $(i = 1, \ldots\ldots, n)$, i. e,

$$\alpha_i (b_h + \nu) \neq (a_i - \eta - 1) \beta_h \tag{1.2}$$
$$(\nu, \eta = 0, 1, \ldots\ldots; h = 1, \ldots\ldots, m; i = 1. \ldots \ldots, n)$$

Also $\xi = \sigma + i\,t$, where σ and t are real. Further the contour L runs from $\sigma - i\infty$ to $\sigma + i\infty$ such that the points :

$$\xi = (b_h + \nu)/\beta_h \; (h=1, \ldots, m; \; \nu=0, 1, \ldots;) \tag{1.3}$$

which are poles of $\Gamma(b_h-\beta_h\,\xi)$, lie to the right and the points :

$$\xi = (a_i-\eta-1)/\alpha_i \; (i=1, \ldots, n; \; \eta=0, 1, \ldots;) \tag{1.4}$$

which are poles of $\Gamma(1-a_i+\alpha_i\,\xi)$, lie to the left of L. Such a contoure is possible on account of (1.2). These assumptions for the H-function will be adhered to throughout this paper.

2. *Properties of the H-function* [6]. The H-function is symmetric in the pairs $(a_1, \alpha_1), \ldots, (a_n, \alpha_n)$, likewise in $(a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p)$, in $(b_1, \beta_1), \ldots, (b_m, \beta_m)$, and in $(b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q)$.

If one of (a_i, α_i) $i=1, \ldots, n$ is equal to one of (b_j, β_j), $(j=m+1, \ldots, q)$ [or one of the (b_h, β_h) $(h=1, \ldots, m)$ is equal to one of the (a_j, α_j) $(j=n+1, \ldots, p)$], then the H-function reduces to one of the lower order, that is, p, q and n (or m) decreases by unity ; we give below one of such reduction formula :

(a)

$$H^{m,n}_{p,q}\left[x \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (a_1, \alpha_1) \end{matrix} \right.\right]$$

$$= H^{m,n-1}_{p-1,q-1}\left[x \left| \begin{matrix} (a_2, \alpha_2), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}) \end{matrix} \right.\right] \tag{2.1}$$

other reduction formulae being similar.

The obvious changes of the variable in the integral (1.1) give us the following relations :

(b)

$$x^{\sigma} H^{m,n}_{p,q}\left[x \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right.\right] \tag{2.2}$$

$$= H^{m,n}_{p,q}\left[x \left| \begin{matrix} (a_1+\sigma\alpha_1, \alpha_1), \ldots, (a_p+\sigma\alpha_p, \alpha_p) \\ (b_1+\sigma\beta_1, \beta_1), \ldots, (b_q+\sigma\beta_q, \beta_q) \end{matrix} \right.\right]$$

(c)

$$H^{m,n}_{p,q}\left[x^{-1} \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right.\right]$$

$$= H^{n,m}_{q,p}\left[x \left| \begin{matrix} (1-b_1, \beta_1), \ldots, (1-b_q, \beta_q) \\ (1-a_1, \alpha_1), \ldots, (1-a_p, \alpha_p) \end{matrix} \right.\right] \tag{2,3}$$

(d)

$$H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_5, \ldots, (b_q, \beta_q) \end{matrix}\right]$$

$$= c\, H^{m,n}_{p,q}\left[x^c \,\middle|\, \begin{matrix} (a_1, c\,\alpha_1), \ldots, (a_p, c\,\alpha_p) \\ (b_1, c\,\beta_1), \ldots, (b_q, c\,\beta_q) \end{matrix}\right] \tag{2.4}$$

where $c > 0$.

(e)

$$H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix} a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1) \ldots, (b_q, \beta_q) \end{matrix}\right]$$

$$= (2\pi)^{(1-N)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\, N^{\mu}$$

$$\times H^{Nm,Nn}_{Np;Nq}\left[x\,N^{-\delta} \,\middle|\, \begin{matrix} \left(\triangle(N, a_1), \frac{\alpha_1}{N}\right), \ldots, \left(\triangle(N, a_p), \frac{\alpha_p}{N}\right) \\ \left(\triangle(N, b_1) \frac{\beta_1}{N}\right), \ldots, \left(\triangle(N, b_q), \frac{\beta_q}{N}\right) \end{matrix}\right] \tag{2.5}$$

where μ, δ and $(\triangle(N, f), \alpha)$ stand for the quantities

$$\sum_1^q (b_j) - \sum_1^p (a_j) + \tfrac{1}{2}p - \tfrac{1}{2}q\,;\ \sum_1^q (\beta_j) - \sum_1^p (\alpha_j)$$

$$\text{and } \left(\frac{f}{N}, \alpha\right)\ \left(\frac{f+1}{N}\alpha\right), \ldots, \left(\frac{f+N-1}{N}, \alpha\right),$$

respectively and N is a positive integer >2.

Proof of (2.5), To derive (2.5), we first express the H-function on the left-hand side of it in terms of Barnes type of integral with the help of (1.1) and then use the following multiplication theorem for gamma function [3, p. 4].

$$\Gamma(N z) = (2\pi)^{\frac{1}{2}(1-N)}\, N^{Nz-\frac{1}{2}} \prod_{j=0}^{N-1} \Gamma\left(z + \frac{j}{N}\right) \tag{2.6}$$

in the integral thus obtained. The result now easily follows with the help of the equations (1.1) and (2.4).

3. *Asymptotic Expansians.* Braaksma [1, p. 278] has shown that the H-function makes sense and defines an analytic function of x in the following two cases :

(i) $\delta > 0, x \neq 0.$

$$\text{where } \delta = \sum_{1}^{q} (\beta_j) - \sum_{1}^{p} (\alpha_j) \qquad \ldots \qquad \ldots \qquad (3.1)$$

(ii) $\delta = 0$, and $0 < |x| < D$.

$$\text{where } D = \prod_{1}^{p} (\alpha_j)^{\alpha_j} \prod_{1}^{q} (\beta_j)^{-\beta_f} \qquad \ldots \qquad \ldots \qquad (3.2)$$

From the equation (6.5) of Braaksma's paper we have ;

$$H^{m, n}_{p, q}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1) \ldots\ldots, (a_p, \alpha_p)\\ (b_1, \beta_1) \ldots \ldots, (b_q, \beta_q)\end{matrix}\right] = O\left((|x|^{\alpha}\right) \text{for smal } x \qquad (3.3)$$

where $\delta \geqslant 0$ and $\alpha = R\,(b_h/\beta_b)\;(h=1, \ldots\ldots, m)$.

The behaviour of the H-function for large x has been considered at full length by Braaksma in the memoir referred to in the beginning of this section. There he has considered different sets of conditions of the convergence of the integral given by (1.1), We shall, however, for lack of space, restrict ourselves throughout this paper to the case when the parameters of the H-function satisfy the following conditions of validity ;

$$\text{(i)} \quad \lambda = \sum_{1}^{n} (\alpha_j) - \sum_{n+1}^{p} (\alpha_j) + \sum_{1}^{m} (\beta_j) - \sum_{m+1}^{q} (\beta_j) > 0 \qquad (3,4)$$

$$\text{(ii)} \quad |\arg x| < \tfrac{1}{2}\lambda\pi \qquad (3.5)$$

Again from the equation (2.16) of the same paper we get :

$$H^{m, n}_{p, q}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_q)\end{matrix}\right] = O\left(|x|^{\beta}\right) \text{for large } x, \qquad (3.6)$$

where $\delta > 0$, the conditions (i) and (ii) given above are satisfied and β stands for

$R\left(\frac{a_i - 1}{\alpha_i}\right)$ $(i=1, \ldots\ldots, n)$.

Finally from the equations (2.36), (2.43) and (4.12) of the same paper, we infer that if $n=0$, the H-function vanishes exponentially for large x in certain cases *i. e*, we have :

$$H^{m, o}_{p, q}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p)\\ (b_1, \beta_1, \ldots \ldots, b_q, \beta_q)\end{matrix}\right] \sim O\left(exp\,\{-\delta\, x^{1/\delta}\, \beta^{1/\delta}\}\, x^{(\frac{1}{2}+\mu)\,/\delta}\right)$$

(3.7)

where μ stands for the quantities $\sum_1^q (b_j) - \sum_1^p (a_j) + \frac{1}{2}p - \frac{1}{2}q$, and the conditions given below are satisfied :

$$\text{(i)} \quad \sum_1^m (\beta_j) - \sum_1^p (\alpha_j) - \sum_{m+1}^q (\beta_j) > 0 \tag{3.8}$$

$$\text{(ii)} \quad |\arg x| < \tfrac{1}{2}\pi \left[\sum_1^m (\beta_j) - \sum_1^p (\alpha_j) - \sum_{m+1}^q (\beta_j) \right] \tag{3.9}$$

$$\text{(iii)} \quad \delta = \sum_1^q (\beta_j) - \sum_1^p (\alpha_j) > 0 \tag{3.10}$$

Remarks. On account of the formula (2.3), we can transform the H-function with $\delta < 0$ and arg (x) to one with $\delta > 0$ and arg $\left(\frac{1}{x}\right)$. Hence the asymptotic expansions for the case $\delta < 0$ can also be obtained from the results of Braaksma, mentioned above by interchanging the role of x at $x = 0$ and $x = \infty$.

4. *Particular cases of the H-function* [6].

(i) On taking $\alpha_j = \beta_h = 1$ ($j = 1, \ldots\ldots, p$; $h = 1, \ldots\ldots, q$) (in (1.1), we have with the help of the definition of the G-function [3, p, 207].

$$H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_1, 1), \ldots\ldots, (a_p, 1)\\(b_1, 1), \ldots\ldots, (b_q, 1)\end{matrix}\right] = G^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}a_1, \ldots\ldots, a_p\\b_1, \ldots\ldots, b_q\end{matrix}\right] \tag{4.1}$$

(ii) Also if we take $\alpha_j = \beta_h = \frac{1}{o}$ ($j = 1, \ldots\ldots, p$; h 1, ..., q) in (2.4) and then replace $\frac{1}{c}$ by c' we get the following result :

$$H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_1, c'), \ldots\ldots, (a_p, c')\\(b_1, c'), \ldots\ldots, (b_q, c')\end{matrix}\right] = 1/c'\, G^{m,n}_{p,q}\left[x^{1/c'} \,\middle|\, \begin{matrix}a_1, \ldots\ldots, a_p\\b_1, \ldots\ldots, b_q\end{matrix}\right] \tag{4.2}$$

where $c' > 0$.

(iii) $H^{1,h}_{h+k,\, l+r}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots\ldots, (a_h, \alpha_h), (b_1, \beta_1), \ldots\ldots, (b_k, \beta_k)\\(c_1, \gamma_1), \ldots\ldots, (c_l, \gamma_l), (d_1, \delta_1), \ldots\ldots, (d_r, \delta_r)\end{matrix}\right]$

$$= (2\pi)^{\frac{1}{2}(h+l-k-r+B+D-A-C)}$$

$$\times \prod_{1}^{h} (\alpha_j)^{\frac{1}{2}-a_j} \prod_{1}^{k} (\beta_j)^{\frac{1}{2}-b_j} \prod_{1}^{l} (\gamma_j)^{c_j-\frac{1}{2}} \prod_{1}^{r} (\delta_j)^{d_j-\frac{1}{2}}$$

$$\times G^{C,\,A}_{A+B,\,C+D}\left[\frac{\prod_{1}^{h}(\alpha_j)^{\alpha_j}\prod_{1}^{k}(\beta_j)^{\beta_j}x}{\prod_{1}^{l}(\gamma_j)^{\gamma_j}\prod_{1}^{r}(\delta_j)^{\delta_j}}\left|\begin{matrix}\{\Delta(\alpha_h, a_h)\}, \{\Delta(\beta_k, b_k)\}\\ \{\Delta(\gamma_l, c_l)\}, \{\Delta(\delta_r, d_r)\}\end{matrix}\right.\right], \quad (4.3)$$

where l, h, k and r are integers satisfying $1 \leqslant l$; $0 \leqslant h$; $0 \leqslant k$; $0 \leqslant r$, and $\alpha_1, \ldots\ldots, \alpha_h$; $\beta_1, \ldots\ldots, \beta_k$; $\gamma_1, \ldots\ldots, \gamma_l$; $\delta_1, \ldots\ldots, \delta_r$ are positive integers and A, B, C, D, $\{\Delta(\alpha_p, a_p)\}$, stands for the quantities mentioned below :

$$A = \sum_{1}^{h}(\alpha_j)\;;\; B = \sum_{1}^{k}(\beta_j)\;;\; C = \sum_{1}^{l}(\gamma_j)\;;\; D = \sum_{1}^{r}(\delta_j)\;;$$

$$\{\Delta(\alpha_p, a_p)\} = \Delta(\alpha_1, a_1), \ldots\ldots, \Delta(\alpha_p, a_p).$$

Proof of (4.3). To obtained (4.3), we write the H-function in it, in terms of Barnes integral with the help of (1.1) and then apply (2.6) in gamma functions occurring in the integrand. The result follows immediately on interpreting the new integrand thus obtained, with the help of the definition of the G-function.

$$\text{(iv)}\quad x^l K_\nu(x) = 2^{l-1} H^{2,0}_{0,2}\left[\frac{x^2}{4}\middle| (\tfrac{1}{2}l - \tfrac{1}{2}\nu, 1), (\tfrac{1}{2}l + \tfrac{1}{2}\nu, 1)\right], \quad (4.4)$$

where $K_\nu(x)$ is a modified Bessel function.

$$\text{(v)}\quad x^l e^{-\frac{1}{2}x} W_{k,r}(x) = H^{2,0}_{1,2}\left[x\left|\begin{matrix}(l-k+1, 1)\\ (l+r+\frac{1}{2}, 1), (l-r+\frac{1}{2}, 1)\end{matrix}\right.\right] \quad (4.5)$$

where $W_{k,r}(x)$ is a Whittaker function.

(vi) ${}_pF_q(a_1, \ldots\ldots, a_p; b_1, \ldots\ldots, b_q; -x)$

$$= \frac{\prod_1^q \Gamma(b_j)}{\prod_1^p \Gamma(a_j)} H^{1,p}_{p,q+1}\left[x \begin{array}{l} (1-a_1, 1), \ldots\ldots, (1-a_p, 1) \\ (0, 1), (1-b_1, 1), \ldots\ldots, (1-b_q, 1) \end{array}\right] \quad (4.6)$$

(vii) ${}_1F_0(\alpha; -x) = (1+x)^{-\alpha} = \frac{1}{\Gamma(\alpha)} H^{1,1}_{1,1}\left[x \,\middle|\, \begin{array}{l} (1-\alpha, 1) \\ (0, 1) \end{array}\right]$

$$(4.7)$$

(viii) $E(a_1, \ldots\ldots, a_p : b_1, \ldots\ldots, b_q, : x)$

$$= H^{p,1}_{q+1,p}\left[x \,\middle|\, \begin{array}{l} (1, 1), (b_1, 1), \ldots\ldots, (b_q, 1) \\ (a_1, 1), \ldots\ldots, (a_p, 1) \end{array}\right] \quad (4.8)$$

The function defined by the series:

$$\sum_{r=0}^{\infty} \frac{\prod_1^p \Gamma(a_j + \alpha_j r)}{\prod_1^q \Gamma(b_j + \beta_j r)} \cdot \frac{(-x)^r}{r!}$$

was considered in detail by Wright [9, p. 287]. We shall call this function as Wright's generalized hypergeometric function and denote it symbolically as ;

$${}_p\psi_q\left[\begin{array}{l} (a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_q) \end{array}; -x\right]$$

The following formula gives us the relationship between this function and the H-function :

(ix) ${}_p\psi_q\left[\begin{array}{l} (a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_a) \end{array}; -x\right]$

$$= H^{1,\,p}_{p,\,q+1}\left[x \,\middle|\, \begin{matrix}(1-a_1,\alpha_1),\ldots\ldots,(1-a_p,\alpha_p)\\ (0,1),(1-b_1,\beta_1),\ldots\ldots,(1-b_q,\beta_q)\end{matrix}\right] \tag{4.9}$$

If all the α's and β's are put equal to unity in (4.9), we get the following result an account of the equation (4.6) :

(x) $${}_p\psi_q\left[\begin{matrix}(a_1,1),\ldots\ldots,(a_p,1)\\ (b_1,1),\ldots\ldots,(b_q,1)\end{matrix};x\right]$$

$$= \frac{\prod_1^p(\Gamma a_j)}{\prod_1^q \Gamma(b_j)}\;{}_pF_q(a_1,\ldots\ldots,a_p;b_1,\ldots\ldots,b_q;x) \tag{4.10}$$

(xi) $$J_\nu^\mu(x) = \sum_{r=0}^{\infty}\frac{(-x)^r}{r!\;\Gamma(1+\nu+\mu r)}$$

$$= H^{1,\,0}_{0,\,2}\left[x \,\middle|\, (0,1),(-\nu,\mu)\right] \tag{4.11}$$

where $J_\nu^\mu(x)$ is Maitland's generalized Bessel function [8, p. 257].

5. The following integral will be evaluated in this section.

$$\int_0^\infty x^{\eta-1}\,H^{m,\,n}_{p,\,q}\left[zx^\sigma \,\middle|\, \begin{matrix}(a_1,\alpha_1),\ldots\ldots,(a_p,\alpha)\\ (b_1,\beta_1),\ldots\ldots,(b_q,\beta_q)\end{matrix}\right]\times$$

$$H^{k,\,f}_{r,\,l}\left[sx \,\middle|\, \begin{matrix}(c_1,\gamma_1),\ldots\ldots,(c_r,\gamma_r)\\ (d_1,\delta_1),\ldots\ldots,(d_l,\delta_l)\end{matrix}\right]dx.$$

$$= \frac{1}{s^\eta}\,H^{m+f,\,n+k}_{p+l,\,q+r}\left[\frac{z}{s^\sigma} \,\middle|\, \begin{matrix}\{(a_n,\alpha_n)\},\{(1-d_l-\eta\,\delta_l,\sigma\,\delta_l)\},\\ \{(b_m,\beta_m)\},\{(1-c_r-\eta\,\gamma_r,\sigma\,\gamma)\},\end{matrix}\right.$$

$$\left.\begin{matrix}(a_{n+1},\alpha_{n+1}),\ldots\ldots,(a_p,\alpha_p)\\ (b_{m+1},\beta_{m+1}),\ldots\ldots,(b_q,\beta_q)\end{matrix}\right]$$

provided that, $R\left(\eta + \sigma \frac{b_h}{\beta_h} + \frac{d_i}{\delta_i}\right) > 0$ $(h=1, \ldots\ldots, m\,;\, i=1, \ldots\ldots, k)$,

$$R \cdot \left[\eta + \left(\frac{c_j - 1}{\gamma_j}\right) + \sigma\left(\frac{a_{h'} - 1}{\alpha_{h'}}\right)\right] < 0 \; (j=1, \ldots\ldots, f;\, h'=1, \ldots\ldots, n),\ \sigma > 0,\ \lambda > 0,$$

$\lambda' > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$ and $|\arg s| < \frac{1}{2}\lambda'\pi$, where λ and λ^1, will stand for the quantities

$$\sum_{1}^{m}(\beta_j) - \sum_{m+1}^{q}(\beta_i) + \sum_{1}^{n}(\alpha_j) - \sum_{n+1}^{p}(\alpha_j),\ \sum_{1}^{k}(\delta_j) - \sum_{k+1}^{l}(\delta_j) + \sum_{1}^{f}(\gamma_j) - \sum_{f+1}^{r}(\lambda_j),$$

and $\{(a_p, \alpha_p)\}$ for the quantities $(a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p)$, throughout this paper.

Remark. We have given above only one set of conditions of validity of (5.1). Remaining sets of conditions of validity can be obtatined by further considering the asymptotic exansion of the H-function given by Braakshma [1].

Proof. Substituting the value of

$$H^{m,n}_{p,q}\left[z\, x^{\sigma} \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots\ldots, (a_p, \alpha^p)\\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_q)\end{matrix}\right]$$

in terms of Barne's integral with the help of (1.1) in the integrand of the equation (5.1), it reduces to

$$\int_0^\infty x^{\eta-1}\left[\frac{1}{2\pi i}\int_L \frac{\prod_{1}^{m}\Gamma(b_j - \beta_j\xi)\prod_{1}^{n}\Gamma(1 - a_j + \alpha_j\xi)}{\prod_{m+1}^{q}\Gamma(1 - b_j + \beta_j\xi)\prod_{n+1}^{p}\Gamma(a_j - \alpha_j\xi)}\, x^{\sigma\xi} z^{\xi}\, d\xi\right.$$

$$\left.\times\, H^{k,f}_{r,l}\left[sx \,\middle|\, \begin{matrix}(c_1, \gamma), \ldots\ldots, (c_r, \gamma_r)\\ (d_1, \delta_1), \ldots\ldots, (d_l, \delta_l)\end{matrix}\right]\right]dx \tag{A}$$

If we now interchange the order of integration in (A) and replace sx by x therein, it reduces to

$$\frac{1}{2\pi i}\int_L \frac{\prod_{1}^{m} \Gamma(b_j - \beta_j \xi) \prod_{1}^{n} \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{m+1}^{q} \Gamma(1-b_j+\beta_j \xi) \prod_{n+1}^{p} \Gamma(a_j-\alpha_j \xi)} z^{\xi}$$

$$\times\left[\frac{1}{s^{\eta+\sigma\xi}}\int_0^{\infty} x^{\eta+\sigma\xi-1}\, H^{k,\,f}_{r,\,1}\left[x \,\middle|\, \begin{matrix}(c_1,\gamma_1),\ \ldots\ldots,(c_r,\gamma_r)\\(d_1,\delta_1),\ \ldots\ldots,(d_1,\delta_1)\end{matrix}\right] d \right] d\xi \tag{B}$$

Since the x-integral in (B) is L $(0,\infty)$ under the conditions stated with (5.1) and the H-function

$$H^{k,\,f}_{r,\,1}\left[x \,\middle|\, \begin{matrix}(c_1,\gamma_1),\ \ldots\ldots,(c_r,\gamma_r)\\(d_1,\delta_1),\ \ldots\ldots,(d_1,\delta_r)\end{matrix}\right]$$

satisfies the remaining conditions of validity of the application of Fourier Mellin inversion theorem [4, p. 307], the value of x-integral in (B), follows directly by applying Fourier Mellin inversion theorem in it and interpreting the result thus obtained with the definition of the H-function given in (1.1), and the expression given by (B) reduces to

$$\frac{1}{s^{\eta}}\cdot\frac{1}{2\pi i}\int_L \frac{\prod_{1}^{m} \Gamma(b_j - \beta_j \xi) \prod_{1}^{n} \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{m+1}^{q} \Gamma(1 - b_j + \beta_j \xi) \prod_{n+1}^{p} \Gamma(a_j - \alpha_j \xi)}$$

$$\times\frac{\prod_{1}^{k} \Gamma(d_j + \eta\,\delta_j + \sigma\,\delta_j\,\xi) \prod_{1}^{n} \Gamma(1 - c_j - \eta\,\gamma_j - \sigma\,\gamma_j\,\xi)}{\prod_{1}^{1} \Gamma(1 - d_j - \eta\,\delta_j - \sigma\,\delta_j\,\xi) \prod_{1}^{r} \Gamma(c_j + \eta\,\gamma_j + \sigma\gamma_j\,\xi)}\left(\frac{z}{s^{\sigma}}\right)^{\xi} d\xi \tag{5.2}$$

Interpreting (5.2) with the help of (1.1) we get the required result.

To justify the inversion of the order of integration in (A) we infer from § 3 that x-integral is absolutely convergent there

if $R(\eta + \frac{d_i}{\delta_i}) > 0$ $(i = 1, \ldots\ldots, k)$, $\lambda' > 0$.

$$R\left(\eta + \frac{c_j - 1}{\gamma_j}\right) < 0 \ (j = 1, \ldots\ldots, f) \text{ and}$$

the resulting integral in it is abolutely convergent, when

$$R\left(\eta + \sigma\frac{b_h}{\beta_h} + \frac{di}{\delta i}\right) > 0 \ (h = 1, \ldots\ldots, m;\ i = 1, \ldots\ldots, k) \text{ and}$$

$$R\left(\eta + \frac{c_j - 1}{\gamma_j} + \sigma\frac{a_h' - 1}{\alpha_h'}\right) < 0 \ (j = 1, \ldots\ldots, f;\ h' = 1, \ldots\ldots, n),$$

$\lambda > 0$, $\lambda' > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$ and $|\arg s| < \frac{1}{2}\lambda'\pi$.

To prove the absolute convergence of ξ—integral in (A), we put $\xi = it$, $z = Re^{i\phi}$ in it, and replace gamma function involving—ξ by gamma function involving $+\xi$, by virtue of the well-known formula :

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

We know that for large ξ, the asymptotic expansion of the gamma function is given by :

$$\log \Gamma(\xi + a) = (\xi + a - \tfrac{1}{2}) \log \xi - \xi + \tfrac{1}{2}\log(2\pi) + O(|\xi|^{-1}).$$

Therefore as $|t| \to \infty$, the absolute value of the integrand in ξ—integral of (A) is asymptotically equal to the product of the following expressions :

$$\exp\{-\tfrac{1}{2}\lambda\pi|t| + \phi t + it(\delta \log t - \log R + k_1)\} \text{ and}$$

$$|t|^{\mu}\{k_2 + O(|t|^{-1})\},$$

where $\lambda = \sum_{1}^{n}(\alpha_j) - \sum_{n+1}^{p}(\alpha_j) + \sum_{1}^{m}(\beta_j) - \sum_{m+1}^{q}(\beta_j)$,

$$\delta = \sum_{1}^{q}(\beta_j) - \sum_{1}^{p}(\alpha_j),\ \mu = \frac{p-q}{2} + \sum_{1}^{q}(b_j) - \sum_{1}^{p}(a_j)$$

and k_1, k_2 are constants independent of $|t|$.

Therefore ξ—integral in (A) is absolutely convergent if the following conditions are satisfied

$$\lambda > 0,\ \phi = |\arg z| < \tfrac{1}{2}\lambda\pi$$

From the above discussion it is clear that x-integral, ξ—integral and the resulting integral in (A), require for their absolutely convergence the conditions mentioned in (5.1), a few of which are not included in (5.1), being easily relaxable.

Hence the interchange of the order of integration in (A) is justified by virtue of De La Valêê Poussin's theorem [2, p. 504]. This completes the proof.

6. *Particular cases.*

(*a*) On taking all $\gamma's$ and $\sigma's$ equal to unity in (5.1), we arrive at the following integral by virtue of (4.1) :

$$\int_0^\infty {}^{\eta-1}\ H_{p,\,q}^{m,\,n}\left[z\ x^{\sigma}\ \middle|\ \begin{matrix}(a_1,\alpha_1),\ \ldots\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ \ldots\ldots,(b_q,\beta_q)\end{matrix}\right]\ G_{r,\,l}^{k,\,f}\left[sx\ \middle|\ \begin{matrix}c_1,\ \ldots\ldots,c_r\\d_1,\ \ldots\ldots,d_l\end{matrix}\right]dx$$

$$=\frac{1}{s^{\eta}}\ H_{p+l,\,q+r}^{m+f,\,n+k}\left[\frac{z}{s^{\sigma}}\ \middle|\ \begin{matrix}\{(a_n,\alpha_n)\},\ \{(1-d_l-\eta,\sigma)\},\ (a_{n+1},\alpha_{n+1}),\ \ldots\ldots,(a_p,\alpha_p)\\\{(b_m;\beta_m)\},\ \{(1-c_r-\eta,\sigma)\},\ (b_{m+1},\beta_{m+1}),\ \ldots\ldots,(b_q,\beta_q)\end{matrix}\right], \qquad (6.1)$$

provided that $R\left[\eta+\sigma\dfrac{b_h}{\beta_p}+d_i\right] > 0\ (h=1,\ \ldots\ldots,m\ ;\ i=1,\ \ldots\ldots,k)$,

$$R\left[\eta+(c_j-1)+\sigma\left(\frac{a_{h'}-1}{\alpha_{h'}}\right)\right] < 0\ (j=1,\ \ldots\ldots,f\ ;\ h'=1,\ \ldots\ldots,n),\ \lambda > 0$$

$\sigma > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$, $\mu > 0$ and $|\arg s| < \frac{1}{2}\mu\pi$
where $\mu=2k+2f-l-r$.

If we take $\eta=\sigma$, $k=h$, $f=l$, $r=q$, $s=r$, $m=\alpha$, $n=\beta$, $p=\gamma$, $q=\delta$, $\sigma=N/S$ and on further taking $\alpha's$ and $\beta's$ equal to unity and replacing c's by $\alpha's$ and d's by $\beta's$ in (6.1), we get an integral earlier given by Saxena [7, p. 401].

(*b*) On taking $k=1$, $r=f$ and replacing l by $l+1$ ia (5.1), it reduces to the following integral by virute of (4.9).

$$\int_0^{\infty} x^{\eta-1} \; H^{m,\, n}_{p,\, q}\left[zx^{\sigma} \left| \begin{matrix} (a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_q) \end{matrix} \right. \right]$$

$$ {}_f\psi_1 \left[\begin{matrix} (c_1, \gamma_1), \ldots\ldots, (c_f, \gamma_f) \\ (d_1, \delta_1), \ldots\ldots, (d_l, \delta_l) \end{matrix} ; -sx \right] dx$$

$$= \frac{1}{s\eta} \; H^{m+f,\, n+1}_{p+l+1,\, q+f} \left[\frac{z}{s\sigma} \left| \begin{matrix} \{ (a_n, \alpha_n) \}, (1-\eta, \sigma), \{ (d_l-\eta\, \delta_l\, \lambda, \sigma\, \delta_l) \} \\ \{ (b_m, \beta_m) \}, \{ (c_f-\eta\, \gamma_f, \sigma\, \gamma_f) \}, \end{matrix} \right. \right.$$

$$\left. \left. \begin{matrix} (a_{n+1}, \alpha_{n+1}), \ldots\ldots, (a_p, \alpha_p) \\ (b_{m+1}, \beta_{m+1}), \ldots\ldots, (b_q, \beta_q) \end{matrix} \right. \right], \tag{6.2}$$

where $R\left(\eta+\sigma \;\frac{b_h}{\beta_h}\right) > 0 \; (h=1, \ldots\ldots, m), \sigma > 0, \lambda > 0,$

$$R\left[\eta+\sigma \;\left(\frac{a_{h'}-1}{\alpha_{h'}}\right) - (c_j/\gamma_f)\right] \angle \; 0 \; (h'=1, \ldots\ldots, n \; ; j=1, \ldots\ldots, f)$$

$$\left[1+ \sum_1^f (\gamma_j) - \sum_1^l (\delta_f) \right] > 0, \; | \arg z | < \tfrac{1}{2} \lambda\pi \text{ and } | \arg s | <$$

$$\left[1+ \sum_1^f (\gamma_f) - \sum_1^l (\delta_f)\right] \frac{\pi}{2}$$

(*c*) On putting all γ's and δ's equal to unity in (6.2), it reduces to the following integral by virtue of (4.10).

$$\int_0^{\infty} x^{\eta-1} \, H^{m,\, n}_{p,\, q}\left[zx^{\sigma} \left| \begin{matrix} (a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_q) \end{matrix} \right. \right]$$

$$ {}_fF_l \; (c_1, \ldots\ldots, c_f \,;\, d_1, \ldots\ldots, d_l \,; -sx) \; dx$$

$$= s^{-\eta} \frac{\prod_{1}^{l} \Gamma(d_f)}{\prod_{1}^{f} \Gamma(c_f)} H^{m+f,\, n+1}_{p+l+1,\, q+f}\left[\frac{z}{s^{\sigma}} \middle| \begin{matrix} \{(a_n, \alpha_n)\}, (1-\eta, \sigma), \{(d_l-\eta, \sigma)\} \\ \{(b_m, \beta_m)\}, \{(c_f-\eta, \sigma)\}, \end{matrix} \right.$$

$$\left. \begin{matrix} (a_{n+1}, \alpha_{n+1}), \ldots\ldots, (a_p, \alpha_p) \\ (b_{m+1}, \beta_{m+1}), \ldots\ldots, (b_q, \beta_q) \end{matrix} \right|, \tag{6.3}$$

where $R\left(\eta+\sigma\frac{b_h}{\beta_h}\right) > 0 \; (h=1, \ldots\ldots, m), \quad \lambda > 0, \sigma > 0,$

$R\left[\eta+\sigma\left(\frac{a_h'-1}{\alpha_h'}\right)-(c_f)\right] < 0 \; (h'=1, \ldots\ldots, n\,;\, j=1, \ldots\ldots, j),$

$(f-l+1) > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$ and $|\arg s| < \frac{1}{2}(f-l+1)\pi$.

(*d*) If we take $j=1$, $l=0$ in (6.3), we get the following integral by virtue of (4.7).

$$\int_0^{\infty} x^{\eta-1}(s+x)^{-\alpha} H^{m,\, n}_{p,\, q}\left[zx^{\sigma} \middle| \begin{matrix} (a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_q) \end{matrix} \right] dx$$

$$= \frac{s^{-\alpha-\eta}}{\Gamma(\alpha)} H^{m+1,\, n+1}_{p+1,\, q+1}\left[zx^{\sigma} \middle| \begin{matrix} (1-\eta,\sigma), (a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p) \\ (\alpha-\eta, \sigma), (b_1,\beta_1), \ldots\ldots, (b_q, \beta_q) \end{matrix} \right], \tag{6.4}$$

where $R\left(\eta+\sigma\frac{b_h}{\beta_h}\right) > 0 \; (h=1, \ldots\ldots, m),$

$R\left[\eta+\sigma\left(\frac{a_h'-1}{\alpha_h'}\right)-\alpha\right] < 0 \; (h'=1, \ldots\ldots, n),$

$\lambda > 0$, $\sigma > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$ and $|\arg s| < \pi$.

(*e*) On putting $k=1, f=\tau=0, l=2, d_1=0, d_2=-\nu, \delta_1=1$ and $\delta_2=\mu$ in (5.1), it reduces the following integral by virtue of the result (4.11).

$$\int_0^\infty x^{\eta-1} J_\nu^\mu(sx)\, H^{m,n}_{p,q}\left[zx^\sigma \middle| \begin{matrix}(a_1,\alpha_1),\ldots\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots\ldots,(b_q,\beta_q)\end{matrix}\right] dx$$

$$= s^{-\eta}\quad H^{m,n+1}_{p+2,q}$$

$$\left[\frac{z}{s^\sigma}\middle|\begin{matrix}\{(a_n,\alpha_n)\},\ (1-\eta,\sigma),\ (1+\nu-\eta\mu,\sigma\mu),\ a_{n+1},(\alpha_{n+1}),\ \ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots\ldots,(b_q,\beta_q)\end{matrix}\right] \tag{6.5}$$

where $R\left[\eta+\sigma\dfrac{b_h}{\beta_h}\right]>0 \qquad (h=1,\ldots\ldots,m).$

$\sigma>0,\ \lambda>0,\ \mu<1,\ |\arg z|<\frac{1}{2}\lambda\pi$ and $|\arg s|<(1-\mu)\dfrac{\pi}{2}$

(*f*) On putting $k=l$, $f=1$, replacing r by $r+1$ and taking all γ's and δ, equal to unity in (5.1), we arrive at the following result by virtue of (4.8).

$$\int_0^\infty x^{\eta-1} H^{m,n}_{p,q}\left[zx^\sigma \middle| \begin{matrix}(a_1,\alpha_1),\ldots\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots\ldots,(b_q,\beta_q)\end{matrix}\right] E\,(d_1,\ldots\ldots,d_l\,;\,c_1,\ldots\ldots,c_r\,;\,sx)\,dx$$

$$= s^{-\eta}\quad H^{m+1,n+1}_{p+1,q+r+1}\left[\frac{z}{s^\sigma}\middle|\begin{matrix}\{(a_n,\alpha_n)\},\{(1-d_l-\eta,\sigma)\},\\ \{(b_m,\beta_m)\},(-\eta,\sigma),\{(1-c_r-\eta,\sigma)\},\end{matrix}\right.$$

$$\left.\begin{matrix}(a_{n+1},\alpha_{n+1}),\ldots\ldots,(a_p,\alpha_p)\\(b_{m+1},\beta_{m+1}),\ldots\ldots,(b_q,\beta_q)\end{matrix}\right], \tag{6.6}$$

where $R\ (\eta+\sigma\dfrac{b_h}{\beta_h}+di)>0\ (h=1,\ldots\ldots,m\,;\,i=1,\ldots\ldots,l),$

$R\left[\eta+\sigma\left(\dfrac{a_{h'}-1}{\alpha_{h'}}\right)\right]<0\ (h'=1,\ldots\ldots,n),\quad \lambda>0,\ \sigma>0,\ |\arg z|<\frac{1}{2}\lambda\pi,$

$\left[1+\sum_1^l(\delta_f)-\sum_1^r(\gamma_f)\right]>0$, and $|\arg s|<\frac{1}{2}\left[1+\sum_1^l(\delta_j)\quad\sum_1^r(\delta_j)\right]\pi.$

(*g*) On taking $k=l=2$, $f=0$, $r=1$, $c_1=1-k$, $d_1+\frac{1}{2}=r$, $d_2=\frac{1}{2}-r$ and $\gamma_1=\delta_1=\delta_2=1$ in (5.1) we get the following integral by virtue of (4.5) :

$$\int_0^\infty x^{\eta+1}\, e^{-\frac{1}{2} sx}\, W_{k,\, r}\,(sx)\; H^{m,\, n}_{p,\, q}\left[\lambda z^{\sigma} \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_q)\end{matrix}\right] dx$$

$$= s^{-\eta}\; H^{m,\, n+2}_{p+2,\, q+1}$$

$$\left[\frac{z}{s^{\sigma}} \,\middle|\, \begin{matrix}(\tfrac{1}{2}+r-\eta, \sigma), (\tfrac{1}{2}-r-\eta, \sigma), (a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p)\\ (b_1\, \beta_1), \ldots\ldots, (b_q, \beta_q)\, (k-\eta, \sigma)\end{matrix}\right], \qquad (6.7)$$

where $R\left[\eta+\sigma\, \dfrac{b_h}{\beta_h}+\tfrac{1}{2}\pm r\right] > 0\; (h=1, \ldots\ldots, m),\; \sigma > 0,$

$\lambda > 0$, $R\,(s) > 0$ and $|\arg z| < \tfrac{1}{2}\lambda\pi$.

(*h*) On taking $k=l=2$, $f=r=0$, $d_1=\tfrac{1}{2}\, v$, $d_2=-\tfrac{1}{2}\, v$, and $\delta_1=\delta_2=\tfrac{1}{2}$ and replacing s by s/2 in (5.1), we get by virtue of (4.4), the following integral due to Gupta [6].

$$\int_0^\infty x^{\eta-1}\; K_{\nu}\,(sx)\; H^{m,\, n}_{p,\, q}\left[zx^{\sigma} \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_q)\end{matrix}\right] dx$$

$$= 2^{\eta-2}\; s^{-\eta}\; H^{m,\, n+2}_{p+2,\, q}$$

$$\left[z\left(\frac{2}{s}\right)^{\sigma} \,\middle|\, \begin{matrix}(1+\tfrac{1}{2}\, \nu-\tfrac{1}{2}\,\eta, \tfrac{1}{2}\,\sigma), (1-\tfrac{1}{2}\, \nu-\tfrac{1}{2}\,\eta, \tfrac{1}{2}\,\sigma), (a_1, \alpha_1), \ldots\ldots, (a_p, \alpha_o)\\ (b_1, \beta_1), \ldots\ldots, (b_q, \beta_q)\end{matrix}\right], \qquad (6{\cdot}8)$$

where $R\left[\eta+\sigma\, \dfrac{b_h}{\beta_h} \pm \nu\right] > 0\; (h=1, \ldots\ldots, m),$

$\lambda > 0$, $\sigma > 0$, $R\,(s) > 0$ and $|\arg z| < \tfrac{1}{2}\lambda\pi$.

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INFINITE SERIES INVOLVING PRODUCT OF TWO FUNCTIONS SATISFYING TRUESDELL F-EQUATIONS

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ABSTRACT

In this paper we will express the infinite series involving the product of two functions satisfying the Truesdell F-equations [5, p. 15], in terms of a double series which under certain circumstances can be sumed up in a compact form. This fact is illustrated by three examples. These results obtained, give us a sum of an infinite series involving the product of two MacRobert E-functions in terms of another E-functions. The first two results have been obtained by Ragab and MacRobort, and the Third appears to be new.

In order to obtain the main result of this paper we will make use of the following operator, [1, p. 250].

$$\nabla (h) = \frac{\Gamma (h)\, \Gamma (\delta + \delta' + h)}{\Gamma (\delta + h)\, \Gamma (\delta' + h)}, \text{ where } \delta = x\frac{d}{dx} \text{ and } \delta' = y\frac{d}{dy} \qquad (1)$$

and

$$\nabla (h) \left[(h)_m (h)_n x^m y^n \right] = (h)_{m+n}\, x^m y^n \qquad (2)$$

We will also use the following results :—

$$x^n D^n = \prod_{\alpha=0}^{n-1} (\delta - \alpha) = (-)^n (-\delta)_n \; [4, \text{p. } 183] \qquad (3)$$

$$\frac{\Gamma (1 - a - n)}{\Gamma (1-a)} = (-)^n/(a)_n \qquad a \neq \text{an integer} \qquad (4)$$

Gauss Theorem

$$ {}_2F_1 \left[\begin{matrix} \alpha, \beta : 1 \\ \gamma \end{matrix} \right] = \frac{\Gamma (\gamma)\, \Gamma (\gamma - \alpha - \beta)}{\Gamma (\gamma - \alpha)\, \Gamma (\gamma - \beta)} \qquad (5)$$

Saalschutz's Theorem.

$$ {}_3E_2 \left[\begin{matrix} -n, \alpha, \beta ; 1 \\ \gamma, 1 + \alpha + \beta - \lambda - n \end{matrix} \right] = \frac{(\gamma - \alpha)_n (\gamma - \beta)_n}{(\gamma)_n (\gamma - \alpha - \beta)_n} \qquad (6)$$

A result due to MacRobet [2, p. 255].

$$ {}_4F_3 \left[\begin{matrix} -n, \alpha, \frac{1}{2}\alpha+1, \beta ; 1 \\ 1/2\,\alpha, \alpha-\beta+1, 2\beta-n+2 \end{matrix} \right] = \frac{(-\beta-1)_n (\alpha-2\beta)n-1}{(\alpha-\beta+1)_n (-2\beta-1)_n} $$

$$ (\alpha - 2\beta - 1 + 2n) \tag{7} $$

Therem :—

If F (x, α), satisfying the Truesuell F-equation then

$$ F(x, \alpha) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \phi(\alpha+n) \tag{8} $$

hence

$$ F(x, \alpha)\, F(y, \beta) = \sum_{m,\, n=0}^{\infty} \frac{(x-x_0)^n}{n!} \frac{(y-y_0)^m}{m!} \phi(\alpha+n)\, \phi(\beta+n) \tag{9} $$

operating by $\nabla(h)$ on both sides of (9), we get

$$ \sum_{r=0}^{\infty} \frac{(x-x_0)^r}{(h)_r} \frac{(y-y_0)^r}{r!} F(x, \alpha+r)\, F(y, \beta+r) = $$

$$ \sum_{m,\, n=0}^{\infty} \frac{(x-x_0)^n}{n!} \frac{(y-y_0)^m}{m!} \cdot \frac{(h)_{m+n}}{(h)_n\,(h)_m} \Phi(\alpha+n)\, \Phi(\beta+m) \tag{10} $$

Similarly for the function $\gamma(x, \alpha)$ satisfying the Truesdell descending F-equation we get

$$ \sum_{r=0}^{\infty} \frac{(x-x_0)^r}{(h)_r} \frac{(y-y_0)^r}{r!} F(x, \alpha-r)\, F(y, \beta-r) $$

$$ = \sum_{m,\, n=0}^{\infty} \frac{(x-x_0)^n}{n!} \frac{(y-y_0)^m}{m!} \frac{(h)_{m+n}}{(h)_n\,(h)_m} \phi(\alpha-n)\, \phi(\beta-m) \tag{11} $$

Cor. :—Let $F(x, \alpha) = e^{\pi i \alpha} E\left(\begin{matrix} p ; a_p + \alpha ; 1/x \\ q ; b_q + \alpha \end{matrix} \right)$

Then using (10) we get when $x_0 = 0$

$$\sum_{r=0}^{\infty} \frac{x^r y^r}{(h)_r\, r!} E\left(\begin{matrix} p\,;\, a_p + \alpha + r\,;\, 1/x \\ q\,;\, b_q + \alpha + r \end{matrix}\right) E\left(\begin{matrix} p'\,;\, a'_{p'} + \alpha + r\,;\, 1/y \\ q'\,;\, b'_{q'} + \alpha + r \end{matrix}\right)$$

$$= \sum_{m=0}^{\infty} \frac{y^m (-)^m}{m!} \frac{\prod_{r=1}^{p} \Gamma(a_r + \alpha + m) \prod_{r=1}^{p'} \Gamma(a'_r + \alpha)}{\sum_{r=1}^{q} \Gamma(b_r + \alpha + m) \prod_{r=1}^{q'} \Gamma(b_r' + \alpha)}$$

$${}_{2+p'+q}F_{1+p+q'} \left[\begin{matrix} -m,\, a'_{p'} + \alpha,\, 1 - h - m,\, 1 - b_q - \alpha - m\,; \\ b'_q + \alpha,\, h,\, 1 - a_p - \alpha - m, \end{matrix} (-)^{(p-q)n} x/y \right] \tag{12}$$

where a_p represents $a_1, a_2 \,\ldots\ldots\ldots, a_p$

Ex. 1.

Let $\alpha = 0$, $x = y$, $h = \gamma$

$p = 2$, $q = 0$; $a_1 = \gamma$, $a_2 = a + \beta - \delta$

$p' = 3$, $q' = 1$; $a'_1 = \gamma$, $a'_2 = \delta - a$, $a'_3 - \delta - \beta$, $b'_1 = \delta$

Using (12) we get,

$$\sum_{r=0}^{\infty} \frac{(x)^{2r}}{(\gamma)_r\, r!} E\,(\gamma + r, a + \beta - \delta + r\,;\, -\,;\, 1/x)$$

$$E\,(\gamma + r, \delta - a + r, \delta - \beta + r\,;\, \delta + r\,;\, 1/x)$$

$$= \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} \frac{\Gamma(\gamma)\,\Gamma(a + \beta - \delta + m)\,\Gamma(\lambda)\,\Gamma(\delta - a)\,\Gamma(\delta - \beta)}{\Gamma(\delta)}$$

$${}_3F_2 \left[\begin{matrix} -m,\, \delta - a,\, \delta - \beta \\ 1 - a - m - \beta + \delta,\, \delta \end{matrix} ; 1 \right]$$

with the help of (6) and replasing x by $1/x$, we get [3, p. 86]

$$\sum_{r=0}^{\infty} \frac{(x)^{-2r}}{\Gamma(\gamma + r)\, r!} E\,(\gamma + r, a + \beta - \delta + r\,;\, x)$$

$$E\,(\gamma + r, \delta - a + r, \delta - \beta + r\,;\, \delta + r\,;\, x)$$

$$= \frac{\Gamma(\delta - a)\,\Gamma(\delta - \beta)\,\Gamma(a + \beta - \delta)}{\Gamma(a)\,\Gamma(\beta)} E\,(a, \beta, \gamma\,;\, \delta\,;\, x) \tag{13}$$

Ex. 2.

Let, $x = y$, $h = a + \alpha$

$p = 2, q = 0, a_1 = a, a_2 = \beta$

$p' = 2, q' = 0, a'_1 = a, a'_2 = \gamma$

From (12) we get

$$\sum_{r=0}^{\infty} \frac{x^{2r}}{(a+\alpha)_r\, r!}\, {}_1E\,(a+\alpha+r, \beta+\alpha+r\,; 1/x)$$

$$E\,(a+\alpha+r, \gamma+\alpha+r, 1/x)$$

$$= \sum_{m=0}^{\infty} \frac{(-x)^m}{m!}\, \Gamma\,(a+m+\alpha)\, \Gamma\,(\beta+m+\alpha)\, \Gamma\,(a+\alpha)\, \Gamma\,(\gamma+\alpha)$$

$${}_2F_1\left[\begin{matrix}-m, \gamma+\alpha \\ 1-\beta-m-\alpha\end{matrix}\,; 1\right]$$

Using (5) and replasing x by $1/x$ we obtain

$$\sum_{r=0}^{\infty} \frac{x^{-2r}}{\Gamma\,(a+\alpha+r)\, r!}\, {}_1E\,(a+\alpha+r, \beta+\alpha+r\,; x)$$

$$E\,(a+\alpha+r, \gamma+\alpha+r\,; x)$$

$$= \beta\,(\gamma+\alpha, \beta+\alpha)\, E\,(a+\alpha\,; \beta+\gamma+2\,\alpha\,; x) \qquad (14)$$

Ex. 3.

Let $\alpha = 0$, $x = y$, $h = a$

$p = 2, q = 0, a_1 = a, a_2 = -1-2\beta$

$p' = 4, q' = 2, a_1' = a, a_2' = \alpha, a_3' = \frac{1}{2}\alpha + 1, a_4' = \beta$

$b_1' = \frac{1}{2}\alpha, b_2' = \alpha - \beta + 1$

Similarly using (7) we get

$$\sum_{r=0}^{\infty} \frac{x^{2r}}{(a)_r\, r!}\, F\,(a+r, -1-2\,\beta+r\,; 1/x)$$

$$E\left(a+r, \alpha+r, \tfrac{1}{2}\alpha+1+r, \beta+r ; \tfrac{1}{2}\alpha+r, \alpha-\beta+1+r ; 1/x\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-x)^m \Gamma(a+m) \Gamma(-1-2\beta+m)}{m!} \frac{\Gamma(\alpha) \Gamma(\tfrac{1}{2}\alpha+1) \Gamma(\beta) \Gamma(a)}{\Gamma(\alpha/2) \Gamma(\alpha-\beta+1)}$$

$$ {}_4F_3\left[\begin{matrix} -m, \alpha, \alpha/_2+1, \beta ; 1 \\ 1/2\,\alpha, \alpha-\beta+1, 2\beta+2-m \end{matrix}\right]$$

$$= \frac{\Gamma(-1-2\beta) \Gamma(\alpha) \Gamma(\tfrac{1}{2}\alpha+1) \Gamma(\beta) \Gamma(a) \Gamma\left(\frac{\alpha-2\beta-1}{2}\right)}{\Gamma(-\beta-1) \Gamma(\alpha-2\beta-1) \Gamma\left(\frac{\alpha-2\beta+1}{2}\right) \Gamma(\alpha/2)}$$

$$E\left(a, -\beta-1, \alpha-2\beta-1, \frac{\alpha-2\beta+1}{2}; \frac{\alpha-2\beta-1}{2} ; 1/x\right) \qquad (15)$$

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NOTE ON INTEGRAL TRANSFORMS

By

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ABSTRACT

Theorems and examples involving integral transforms are given.

Introduction :

The Laplace, Meijer, Varma, Stieltze's, Hankel and H-transforms given by

$$\phi_1(p) = p \int_0^\infty e^{-pt} f_1(t)\, dt. \tag{1}$$

$$\phi_2(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p \int_0^\infty (pt)^{\frac{1}{2}}\; K_v(pt) f_2(t)\, dt. \tag{2}$$

$$\phi_3(p) = p \int_0^\infty e^{-pt/2}\; (pt)^{m-\frac{1}{2}}\; W_{k,m}(pt) f_3(t)\, dt. \tag{3}$$

$$\phi_4(p) = p \int_0^\infty \frac{f_4(t)}{(p+t)^\lambda}\, dt. \tag{4}$$

$$\phi_5(p) = p \int_0^\infty (pt)\; J_v\;(pt) f_5(t)\, dt. \tag{5}$$

$$\phi_6(p) = p \int_0^\infty (pt)^{\frac{1}{2}}\; H_v(pt) f_6(t)\, dt. \tag{6}$$

will be symbolically represented by

$$\phi_1(p) \doteq f_1(t)\; ; \; \phi_2(p) \underset{v}{\overset{K}{=}} f_2(t)\; ; \; \phi_3(p) \underset{k,m}{\overset{V}{=}} f_3(t)\; ;$$

$$\phi_4(p) \overset{S}{\underset{\lambda}{=}} f_4(t)\,;\ \phi_5(p) \overset{J}{\underset{v}{=}} f_5(t)\,;\ \phi_6(p) \overset{H}{\underset{v}{=}} f_6(t):$$

hereafter. Equations (2) and (3) reduce to (1) by taking $v=+1/2$ and $k+m=1/2$. The following theorems have been obtained.

Theorem. 1. If $\phi(p) \doteq f(t)$ (7)

and

$$\psi(p) \overset{K}{\underset{v}{=}} t^{2k-\frac{1}{2}} f(t^2) \tag{8}$$

then

$$p^{v/2+1/4} \sqrt{\pi}\, \psi(2\sqrt{p}) \overset{V}{\underset{k,\frac{v}{2}}{=}} t^{k-\frac{v+1}{2}} \phi\left(\frac{1}{t}\right) \tag{9}$$

where the Laplace transform of $|\,f(t)\,|$, Meijer transform of $|\,t^{2k-\frac{1}{2}} f(t^2)\,|$

and Varma transform of $|\,t^{k-v/2-\frac{1}{2}}\, \phi\left(\frac{1}{t}\right)\,|$ exist.

Proof: Applying Goldstein theorem (justified under the conditions of the theorem to the operational pair (1, p. 217)

$$2\, a^{\frac{1}{2}}\, p^{k+\frac{1}{2}}\, K_{2\mu}\left(2\sqrt{ap}\right) \doteq t^{-k}\, e^{-a/2t}\, W_{k,\mu}\left(\frac{a}{t}\right) \tag{10}$$

and $$\phi(p) \doteq f(t) \tag{11}$$

where R $(a) > 0$, R $(p) > 0$, reducing in the usual manner we arrive at (9).

Example 1. Let $f(t) = t^{\lambda-1}(t+a)^{-\lambda-\frac{1}{2}}$

then (1, p. 139) given

$$\phi(p) = 2^{\lambda}\, \Gamma(\lambda)\, a^{-\frac{1}{2}}\, p\, e^{ap/2}\, D_{-2\lambda}\left(\sqrt{2ap}\right)$$

where [R $(\lambda) > 0$, $|\arg a| < \pi$, R $(p) \geqslant 0$] and (2, p. 128) after a little simplification gives

$$\psi(p) = \pi^{-\frac{1}{2}} \left[\left(\frac{p}{2}\right)^{3/2} \frac{a^{k-1}}{\Gamma(\lambda+\frac{1}{2})} \left\{ f(v) + f(-v) \right\} + \left(\frac{p}{2}\right)^{7/2-2k} \Gamma\left(k-1 \pm \frac{v}{2}\right) \right.$$

$$\left. {}_1F_2 \left\{ \begin{matrix} \lambda+\frac{1}{2} \\ \mp\frac{v}{2}-k+2 \end{matrix} ; -\frac{ap^2}{4} \right\} \right]$$

where

$$f(v) = \left(\frac{p}{2}\right)^v a^{v/2} \Gamma(-v) \Gamma\left(k+\lambda+\frac{v}{2}-\frac{1}{2}\right) \Gamma\left(\frac{-v}{2}-k+1\right)$$

$${}_1F_2 \left\{ \begin{matrix} \lambda+\frac{v}{2}-\frac{1}{2}+k \\ k+\frac{v}{2},\ 1+v \end{matrix} ; -\frac{ap^2}{4} \right\}$$

$\Gamma\left(k-1 \pm \frac{v}{2}\right) = \Gamma\left(k-1+\frac{v}{2}\right) \Gamma\left(k-1-\frac{v}{2}\right)$ and $\pm\epsilon$ stands for $+\epsilon$, $-\epsilon$

when ever used in ${}_pF_q$'s.

and R $(a) > 0$, R $(p) > 0$, R $(2k+2\lambda-1) > |$R $(v)|$. Applying (9) we get

$$2^{-\lambda} a^{\frac{1}{2}} p^{a/2+1} [\Gamma(\lambda)]^{-1} \left[a^{k-1} \{ \Gamma(\lambda+\tfrac{1}{2}) \}^{-1} \{ f(v) + f(-v) \} + p^{1-k} \right.$$

$$\left. \Gamma\left(k-1 \pm \frac{v}{2}\right) {}_1F_2 \left[\begin{matrix} \lambda+\frac{1}{2} \\ \mp\frac{v}{2}-k+2 \end{matrix} ; -ap \right] \right]$$

$$\underset{k,\frac{v}{2}}{\overset{V}{=}} t^{k-v+3/2} \; e^{a/2t} \; D_{-2\lambda} \left(\sqrt{\frac{2a}{t}}\right).$$

where R $(k) < \frac{1}{2}$, R $(2k+2\lambda \pm v) > 1$, R $(a) > 0$, R $(p) > 0$.

Tkeorem 2. If $\phi(p) \underset{v}{\overset{H}{=}} f(t)$ (12)

and

$$\psi(p) \underset{\mu}{\overset{k}{=}} t^{\sigma-2} \phi(t) \tag{13}$$

then

$$\pi^{\frac{1}{2}} \; 2^{-\sigma+\frac{1}{2}} \; p^{\sigma+v+\frac{1}{2}} \; \psi(p) =$$

$$\int_0^\infty t^{v+3/2} \, E\left(1, \frac{1+v+\sigma\pm\mu}{2} \; ; \; 3/2, \, v+3/2 : \frac{p^2}{t^2}\right) f(t) \, dt. \tag{14}$$

where H-transform of $|f(t)|$, Meijer transform of $|t^{\sigma-2} \phi(t)|$ exist and (14) is uniformly convergent.

Proof : Applying Goldstien theorem (justified under the conditions of the theorem) to the operational pair $\phi(p) \underset{v}{\overset{H}{=}} f(t)$ and (2, p. 165)

$$2^{\sigma} \; \pi^{-\frac{1}{2}} \; a^{-v-\sigma-2} \; p^{v+5/2} \, \Gamma\left(1+\frac{v+\sigma\pm\mu}{2}\right) \left[\Gamma(v+3/2)\right]^{-1}$$

$${}_3F_2 \left\{ \begin{matrix} 1, \, \frac{1+v+\sigma\pm\mu}{2} \\ 3/2, \, v+3/2 \end{matrix} ; -\frac{p^2}{a^2} \right\}$$

$$\underset{v}{\overset{\mathrm{H}}{=}}\ t^{\sigma-\frac{1}{2}}\,k_{\mu}\,(at)$$

where R $(a) > 0$, R $(\sigma+v) > |\,\mathrm{R}\,(\mu)\,| -2$, reducing in the usual manner we arrive at (14).

Example 2. Take $f(t) = t^{\sigma-5/2}\,\mathrm{K}_{\lambda}\,(at)\ \mathrm{K}_{w}\,(at)$ then by (2, p. 166) we have

$$\phi\,(p) = \frac{2^{\sigma-4}\,\Gamma\left(\frac{\sigma+v}{2}\right)\Gamma\left(\frac{\sigma+v+1}{2}\right)p^{v+5/2}}{a^{\sigma+v}\ \Gamma\,(\sigma+v)}$$

$$\mathrm{E}\left(\frac{1,\ \sigma+v\pm\lambda\pm\omega}{2};\ 3/2,\ v+3/2,\ \frac{\sigma+v}{2},\ \frac{\sigma+v+1}{2}\ :\ \frac{4a^2}{p^2}\right)$$

where R $(a) > 0$, R $(\sigma+v) > |\,\mathrm{R}\,(\lambda)\,| + |\,\mathrm{R}\,(\omega)\,|$, then

$$\psi\,(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{(2a)^{\sigma+v}\ 2^{-\sigma+4}\ \Gamma\,(\sigma+v)}{p^{\sigma+v}\ \Gamma\left(\frac{\sigma+v}{2}\right)\Gamma\left(\frac{\sigma+v+1}{2}\right)}$$

$$\mathrm{E}\left(\begin{matrix}1,\ \sigma+v\pm\lambda\pm\omega,\ \sigma+v+2\pm\mu; \\ 3/2,\ v+3/2,\ \frac{\sigma+v}{2},\ \frac{\sigma+v+1}{2}\end{matrix}\ :\ a^2\,p^2\right)$$

where R $(\sigma+v+2) > |\,\mathrm{R}\,(\mu)\,|$, R $(a) > 0$, R $(p) > 0$, then by virtue of (14) we have

$$\frac{(2a)^{\sigma+v}\ \Gamma\,(\sigma+v)}{\Gamma\left(\frac{\sigma+v}{2}\right)\Gamma\left(\frac{\sigma+v+1}{2}\right)2^{2\sigma-5}}\ \mathrm{E}\left(\begin{matrix}\frac{1,\ \sigma+v\pm\lambda\pm\omega}{2},\ \frac{\sigma+v+2\pm\mu}{2}; \\ 3/2,v+3/2,\ \frac{\sigma+v}{2},\ \frac{\sigma+v+1}{2}\ :\end{matrix}\ a^2\,p^2\right)$$

$$= \int_0^\infty t^{\sigma+v+1}\,\mathrm{K}_{\lambda}\,(at)\ \mathrm{K}_{\omega}\,(at)\ \mathrm{E}\left(\frac{1,\ 1+v+\sigma\pm\mu}{2};\ 3/2,\ v+3/2\ :\ p^2\,t^{-2}\right)dt.$$

where $R(a) > 0$, $R(\sigma+v\pm\lambda+\omega) > 0$, $R(p) > 0$.

Theorem 3. If $\phi(p) \doteq f(t)$ (15)

$$\text{and } \psi(p) \underset{v}{\overset{J}{=}} t^{-2\lambda-5/2} f\left(\frac{1}{t}\right) \tag{16}$$

then

$$2^{-2\lambda-1}\pi^{-\frac{1}{2}} p^{-3/2}\psi(p) =$$

$$\int_0^\infty t^{-2\lambda-2} S_1\left(\frac{v-1}{2}, \frac{-v-1}{2}, \lambda, \lambda+\tfrac{1}{2}; \frac{pt}{4}\right)\phi(t)\,dt \tag{17}$$

where the Laplace transform of $|f(t)|$, Hankel transform of $|t^{-2\lambda-5/2} f\left(\frac{1}{t}\right)|$ exist and (17) is uniformly convergent.

Proof; Applying Goldstien theorem to the operational pair $\phi(p) \doteq f(t)$

and (1, p. 221) $2^{-2\lambda-1}\pi^{-\frac{1}{2}} p^{2\lambda+1} J_{2v}(4a/p) \doteq t^{-2\lambda-1}$

$S_1(v-\frac{1}{2}, -v-\frac{1}{2}, \lambda, \lambda+\frac{1}{2} : at)$

where $R(p) > 0$, $R(v-\lambda) > 0$, reducing in the usual manner we arrive at (17).

Example 3. Take $f(t) = t^{\mu} J_\rho(t)$ then (1, p. 182) gives

$$\phi(p) = \Gamma(\mu+\rho+1)\,p.\,(p^2+1)^{-\mu/2-1/2} P_\mu^{-\rho}\left[p/(p^2+1)^{\frac{1}{2}}\right]$$

where $R(\mu+\rho) > -1$, $R(p) > 1$ and (2, p. 58) after simplification gives

$$\psi(p) = \frac{\pi}{2}\csc\left[\tfrac{1}{2}(\rho-v+2\lambda+\mu+1)\pi\right] p^{v+3/2}$$

$$\times\left[A\ {}_0F_3\left(1+v, \frac{v\mp\rho-2\lambda-\mu-1}{2}+1; \frac{p^2}{16}\right)\right.$$

$$\left.-p^{\rho} B\ {}_0F_3\left(1+\rho, 1+\frac{\rho+2\lambda+\mu+1\pm v}{2}; p^2/16\right)\right]$$

where $-3/2-R(v) < R(-2\lambda-\mu-1) < R(\rho)+3/2$ and

$$A^{-1} = 2^{\rho-2\lambda-\mu-1}\,\Gamma(1+v)\,\Gamma[1+\tfrac{1}{2}(-2\lambda-\mu-1+v\pm\rho)]$$

$$B^{-1} = 2^{2\rho+2\lambda+\mu+1}\,\Gamma(1+\rho)\,\Gamma[1+\tfrac{1}{2}(\rho+2\lambda+\mu+1\pm v)]$$

then

$$2^{-2\lambda-2}\,\pi^{\frac{1}{2}}\,p^{v}\csc[\tfrac{1}{2}(\rho-v+2\lambda+\mu+1)\pi]\,[\Gamma(\mu+\rho+1)]^{-1}$$

$$\times\Big[A\,{}_0F_3\Big(1+v,\frac{v\mp\rho-2\lambda-\mu-1}{2}+1;p^2/16\Big)$$

$$-p^{\rho}\,B\,{}_0F_3\Big(1+\rho,\frac{1+\rho\mp2\lambda+\mu+1\pm v}{2};p^2/16\Big)\Big]$$

$$=\int_0^\infty t^{-2\lambda-1}\,S_1\Big(\frac{v-1}{2},\frac{-v-1}{2},\lambda,\lambda+\tfrac{1}{2};\frac{pt}{4}\Big)(t^2+1)^{-\mu/2-\frac{1}{2}}$$

$$P_\mu^{-\rho}\Big[t\,(t^2+1)^{-\frac{1}{2}}\Big]\,dt$$

where $2R(\lambda) < R(1+\mu-v)$, $R(p) > 0$.

Theorem 4. If $\phi(p) \underset{\rho}{\overset{S}{=}} f(t)$ (18)

and $\psi(p) \underset{v}{\overset{J}{=}} t^{v-3/2}\,\phi(t^2)$ (19)

then

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{2^{\rho}\,\Gamma(\rho)}{p^{\rho-1}}\,\psi(p)\underset{v-\rho+1}{\overset{K}{=}} t^{v-\rho+3/2}\,f(t^2) \qquad (20)$$

where the Stieltze' transform of $|f(t)|$, Hankel transform of $|t^{v-3/2}\,\phi(t^2)|$ and Meijer transform of $|t^{v-\rho+3/2}\,f(t^2)|$ exist.

Proof: Applying Goldstien theorem in the operational pair $\phi(p) \underset{\rho}{\overset{S}{=}} f(t)$

and (2, p. 235) $t^{v/2}\,J_v(at^{1/2}) \underset{\rho}{\overset{S}{=}} \dfrac{a^{\rho-1}}{2^{\rho}\,\Gamma(p)}\,p^{v-\rho/2+3/2}\,K_{v-\rho+1}(ap^{1/2})$

where $a > 0$, $R(v) > -1$, $R(k) > \frac{1}{2} R(v) + \frac{1}{4}$, reducing in the usual manner we arrive at (20).

Example 4. Take $f(t) = t^{k+\rho-2} e^{-at/2} W_{k,\mu}(at)$ then (2, p. 237) gives

$$\phi(p) = \Gamma(k+\rho-\tfrac{1}{2}\pm\mu)\,[\Gamma(\rho)]^{-1} p^{k} e^{ap/2} W_{1-k-\rho}(ap)$$

where $R(\rho) > |R(\mu)| - R(k) + \frac{1}{2}$, $R(a) > 0$, $|\arg p| < \pi$. Then we have

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{2^{\rho}\,\Gamma(\rho)}{p\rho^{-1}}\,\psi(p) = \left(\frac{2}{p}\right)^{2k+v+\rho} p^{5/2}\,(2\pi)^{-\frac{1}{2}}$$

$$G^{22}_{23}\left(\frac{p^2}{4a}\,\middle|\,\begin{matrix}\frac{1}{2}-\mu,\ \frac{1}{2}+\mu\\ k+v,\ -1+k+\rho,\ k\end{matrix}\right)$$

where $p > 0$, $|\arg a| < \pi$, $2R(\rho) > 3/2 + R(v) > R(l-k\pm\mu)$, then applying the theorem we get

$$\left(\frac{2}{p}\right)^{2k+v+\rho} p^{5/2}\,(2\pi)^{-1/2}\,G^{22}_{23}\left(\frac{p^2}{4a}\,\middle|\,\begin{matrix}\frac{1}{2}-\mu,\ \frac{1}{2}+\mu\\ k+v,\ -1+k+\rho,\ k\end{matrix}\right)$$

$$\underset{v-\rho+1}{\overset{K}{=}}\ t^{2k+v+\rho+5/2}\,e^{-at^2/2}\,W_{k,\mu}(at^2)$$

Where $R(a) > 0$, $R(p) > 0$, $R(k+\rho) > R(\mu+\frac{1}{2}) > -R(k+v)$.

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ON SOME SERIES OF WHITTAKER TRANSFORM

By

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ABSTRACT

In this paper, some series relations of Whittaker transform and Laguerre transform—a special case of Whittaker transform have been obtained by using the properties of Whittaker function and Generalised Laguerre polynomial. The same have been used to derive certain series of Meijer's G-function and hypergeometric function ${}_2F_1$.

1. The Laplace transform

$$(1.1)\quad \psi(s) = s\int_0^\infty e^{-st} f(t)\,dt, \quad (\mathrm{R}(s) > 0)$$

has been generalised by Varma (15, p. 17) in the form

$$(1.2)\quad \phi(s) = s\int_0^\infty (2st)^{-\frac{1}{2}}\, W_{k,m}(2st)\, f(t)\,dt, \qquad (\mathrm{R}(s) > 0)$$

which reduces to (1.1) when $k=\frac{1}{2}$ and $m=\pm\frac{1}{2}$; and is known as Whittaker transform.

Further, for $k=\frac{1}{2}+\frac{1}{2}\,l+n$ and $m=\pm\frac{1}{2}\,l$, n being a positive integer, (1.2) reduces to

$$(1.3)\quad \mathrm{L}(s) = (-)^n\, n!\, s\int_0^\infty (2st)^{\frac{1}{2}l+\frac{1}{2}}\, e^{-st}\, \mathrm{L}_n^l(2st)\, f(t)\,dt,$$

where $\mathrm{L}_n^l(z)$ is Generalised Laguerre polynomial,

and is known as L_n^l — transform. We shall represent (1.1), (1.2) and (1.3) symbolically as

$$\psi(s) \doteq f(\mathrm{t}),\ \phi(s\,;k,m) \overset{k}{\underset{m}{=\!=}} f(t) \text{ and}$$

$$L(s\,;l,n) \quad \frac{\frac{1}{2}+\frac{1}{2}l+n}{\pm\frac{1}{2}\,l}\, f(t) \text{ respectively.}$$

The object of this paper is to obtain some series of the transforms defined by (1.2) and (1.3) by using the properties of Whittaker function and Generalised Laguerre polynomial. The same have been used to derive some series of Meijer's G-function and hypergeometric function ${}_2F_1$.

2. THEOREM 1.

If

$$\phi(s\,;k,m\,;\lambda) \quad \overset{k}{\underset{m}{==}} \quad t^{\lambda} f(t)$$

and

$$\psi(s\,;\lambda) \doteqdot t^{\lambda} f(t)$$

then

$$\sum_{r=0}^{\infty} \frac{1}{r\,!}\left(\tfrac{1}{2}-k+m\,;r\right)\left(\tfrac{1}{2}-k-m\,;r\right)\phi(s\,;k-r,m\,;\lambda)$$

$$= (2s)^{k-\frac{1}{4}}\,\psi\left(s\,;\lambda+k-\tfrac{1}{4}\right);$$

provided $R\left(\mu+\lambda+k+\frac{3}{4}\right) > 0$ where $f(t) = 0\,(t^{\mu})$ for small t, $t^{\lambda+k+\frac{3}{4}}\,e^{-st} f(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for $R(s) \geqslant s_0 > 0$; $f(t)$ is continuous for $t \geqslant t_0 > 0$ and the series on the left hand side converges.

Proof :— We have

$$\phi(s\,;k-r,m\,;\lambda) = s\int_0^{\infty} (2st)^{-\frac{1}{4}}\, W_{k-r,\,m}(2st)\, t^{\lambda} f(t)\, dt.$$

Multiplying both the sides by $\frac{1}{r\,!}\left(\frac{1}{2}-k+m\,;r\right)\left(\frac{1}{2}-k-m\,;r\right)$, summing for r from $r=0$ to $r=\infty$ and using the relation due to MacRobert and Ragab (11),

$$\sum_{r=0}^{\infty} \frac{1}{r\,!}(\tfrac{1}{2}-k+m\,;\,r)\;(\tfrac{1}{2}-k-m\,;\,r)\;W_{k-r,m}\;(z) = z^k\,e^{-\frac{1}{2}z},$$

we get

$$\sum_{r=0}^{\infty} \frac{1}{r\,!}\,(\tfrac{1}{2}-k+m;\,r.)\;\tfrac{1}{2}-k-m\,;\,r)\;\phi\;(s\,;\,k-r,\,m\,;\,\lambda)$$

$$= (2s)^{k-\frac{1}{4}}\,s\int_0^{\infty} e^{-st}\;t^{\lambda+k-\frac{1}{4}}\;f\,(t)\,dt$$

$$= (2s)^{k-\frac{1}{4}}\;\psi\;(s\,;\,\lambda+k-\tfrac{1}{4}).$$

Regarding the change of order of integration and summation, we see that (4, p. 500).

(i) the series $\sum_{r=0}^{\infty} \frac{1}{r\,!}\;(\tfrac{1}{2}-k+m\,;\,r)\;(\tfrac{1}{2}-k-m\,;\,r)\;W_{k-r,\,m}\;(2st)$

is uniformly convergent in any interval $0 \leqslant t \leqslant a$, a being arbitrary ;

(ii) $f\,(t)$ is continuous for all values of $t \geqslant t_0 > 0$

and

(iii) $\int_0^{\infty} e^{-st}\,\left|\,t^{\lambda+k-\frac{1}{4}}\;f\,(t)\,\right|\,dt$ is convergent, if,

$R\;(\mu+\lambda+k+\tfrac{3}{4}) > 0$, where $f\,(t) = 0\;(t^{\mu})$ for small t and $t^{\lambda+k+\frac{3}{4}}\;e^{-st}\;\;f\,(t) \to 0$ as $t \to \infty$ for $R\;(s) \geqslant s_0 > 0$.

(2.1) Example.

Let $f\,(t) = e^{-qt}$, then (2, p. 13),

$$(2.1)\quad t^{\lambda}\, e^{-qt} \underset{m}{\overset{k}{=\!=}} \frac{\Gamma(\lambda \pm m + 5/4)}{2\;(2s)^{\lambda}\;\Gamma(\lambda - k + 7/4)} \times$$

$$\times\; {}_2F_1(\lambda + m + 5/4,\ \lambda - m + 5/4\ ;\ \lambda - k + 7/4\ ;\ \tfrac{1}{2} - \tfrac{1}{2}\, q/s) = \phi(s\ ;\ k,\ m\ ;\ \lambda)$$

$R(\lambda \pm m + 5/4) > 0$, $R(s) > 0$ and $|s| > |q|$.

Also (9, p. 144)

$$(2.2)\quad t^{\lambda}\, e^{-qt} \doteq s\, \Gamma(\lambda+1)\ (s+q)^{-(\lambda+1)} = \psi(s\ ;\ \lambda)$$

$R(\lambda + k + \tfrac{3}{4}) > 0$ and $R(s+q) > 0$.

Using (2.1) and (2.2) in Theorem 1, and replacing $\lambda + m + 5/4$ by a, $\lambda - m + 5/4$ by b, $\lambda - k + 7/4$ by c and $\tfrac{1}{2} - \tfrac{1}{2}\, q/s$ by z, we get

$$\sum_{n=0}^{\infty} \frac{(c-a\ ;\ r)\,(c-b\ ;\ r)}{r\,!\,(c\ ;\ r)}\ {}_2F_1(a,\ b\ ;\ c+r\ ;\ z)$$

$$= \frac{\Gamma(c)\quad \Gamma(a+b-c)}{\Gamma(a)\quad \Gamma(b)}\ .\ (1-z)^{c-a-b}$$

$R(a) > 0$, $R(b) > 0$, $R(a+b-c) > 0$ and $|z| < 1$.

3. THEOREM 2.

If

$$\phi(s\ ;\ k,\ m\ ;\ \lambda) \underset{m}{\overset{k}{=\!=}} t^{\lambda} f(t)$$

then

$$\sum_{r=0}^{n} \frac{(-)^{n+r} \binom{n}{r} (2s)^{r}\, \Gamma(\beta+n+r)}{\Gamma(1+\beta+2r)}\ \phi(s\ ;\ \tfrac{1}{2} - m,\ m+r\ ;\ \lambda + r)$$

$$= \frac{1}{(\beta+2n)\,(2s)^{\frac{1}{2}\beta}}\ \phi(s\ ;\ \tfrac{1}{2} - m + \tfrac{1}{2}\,\beta + n,\ m - \tfrac{1}{2}\,\beta\ ;\ \lambda - \tfrac{1}{2}\,\beta)$$

provided the Whittaker transforms involved exist.

Proof :—We have

$$\phi\left(s\,;\tfrac{1}{2}-m,\,m+r\,;\,\lambda+r\right)=s\int_0^\infty (2st)^{-\frac{1}{4}}\,W_{\frac{1}{2}-m,\;m+r}(2st)\;t^{\lambda+r}f(t)\,dt.$$

Multiplying both the sides by $\dfrac{(-)^{n+r}\binom{n}{r}(2s)^r\,\Gamma(\beta+n+r)}{\Gamma(1+\beta+2r)}$,

summing for r from $r=0$ to $r=n$ and using the result due to the author (8),

$$\sum_{r=0}^{n}\frac{(-)^{n+r}\binom{n}{r}\Gamma(\beta+n+r)}{\Gamma(1+\beta+2r)}\cdot z^r\,W_{\frac{1}{2}-m,\;m+r}(z)=$$

$$=\frac{W_{\frac{1}{2}-m+\frac{1}{2}\beta+n,\;m-\frac{1}{2}\beta}(z)}{(\beta+2n)z^{\frac{1}{2}\beta}},$$

we get the result.

3.1. Example :—

Using (2.1) in theorem 2 and replacing $\lambda+m+5/4$ by a, $\lambda-m+5/4$ by b and $\frac{1}{2}(1-q/s)$ by z, we get after a little simplification,

$$\sum_{r=0}^{n}\frac{(-n\,;r)\;\Gamma(\beta+n+r)\;\Gamma(a+2r)}{r!\;\Gamma(1+\beta+2r)\;\Gamma(a+r)}\cdot{}_2F_1(a+2r,b\,;a+r\,;z)$$

$$=\frac{(1-a+\beta\,;n)}{\beta+2n}\cdot{}_2F_1(a-\beta,b\,;a-\beta-n\,;z)$$

$R(a)>0$, $R(b)>0$, $R(a-\beta)>0$ and $|z|<1$.

With $z=0$ and a replaced by $a+1$, it yields a known result due to Carlitz (5, p. 90).

4. THEOREM 3.

If

$$\phi(s\,;k,m\,;\lambda)\overset{k}{\underset{m}{=}}t^\lambda f(t)$$

then

$$\sum_{r=0}^{n} \frac{(-)^{n+r}\,(2s)^{\frac{1}{2}r}}{r!\,(n-r)!}\,\phi\,(s\,;\,k-\tfrac{1}{2}r,\ m+\tfrac{1}{2}r\,;\,\lambda+\tfrac{1}{2}r)$$

$$= \frac{\Gamma\,(m+k+\frac{1}{2})}{n!\,\Gamma(m+k-n+\frac{1}{2})}\,\phi\,(s\,;\,k-n,\ m\,;\ \lambda)$$

provided R $(\frac{1}{2}-k+m) > 0$ and the Whittaker transforms involved exist.

Proof :—We have

$$\phi\,(s\,;\,k-\tfrac{1}{2}r,\ m+\tfrac{1}{2}r\,;\,\lambda+\tfrac{1}{2}r)$$

$$= s\int_0^{\infty} (2st)^{-\frac{1}{4}}\ W_{k-\frac{1}{2}r,\ m+\frac{1}{2}r}\ (2st)\ t^{\lambda+\frac{1}{2}r}\,f\,(t)\,dt.$$

Multiplying both the sides by $(-)^{n+r}\ (2s)^{\frac{1}{2}r}/\,r!\ (n+r)\,!$

and summing for r from $r=0$ to $r=n$ and using (14, p. 54)

$$W_{k-n,\,m}\ (z) = \frac{(-)^n\ n!\ \Gamma\,(m+k-n+\frac{1}{2})}{\Gamma\,(m+k+\frac{1}{2})}\ \sum_{r=0}^{n}\ \frac{(-)^r\ \ z^{\frac{1}{2}r}W_{k-\frac{1}{2}r,\ m+\frac{1}{2}r}\ (z)}{r!\ \ (n-r)\,!}$$

we get the result.

5. Here, we state a theorem established earlier by Bose (3, p. 210) in a slightly different form.

Theorem 4.

If

$$\phi\,(s\,;\,k,\ m\,;\lambda\,)\ \underset{m}{\overset{k}{==}}\ t^{\lambda}\ f\,(t)$$

then

$$\sum_{r=0}^{n}\ \frac{(-)^{n+r}\ \ (2s)^{\frac{1}{2}r}}{r!\,(n-r)!\ \Gamma\,(m+k+r+\frac{1}{2})}\ \phi\,(s\,;\,k+\tfrac{1}{2}\,r,\ m+\tfrac{1}{2}\,r\,;\,\lambda+\tfrac{1}{2}\,r)$$

$$= \frac{1}{n\,!\,\Gamma\,(m+k+n+\frac{1}{2})}\,\phi\,(s\,;\,k+n,\,m\,;\,\lambda)$$

provided R $(\frac{1}{2}-k+m) > 0$ and the Whittaker transforms involved exist.

5.1. Examples :—

Let $\quad t^{\lambda}\,f\,(t) = t^{\lambda-1}\,G^{l-2,u-1}_{p-1,\,q-2}\left[\frac{1}{t}\middle|\begin{matrix} a_2, \ldots\ldots\ldots\ldots, a_p \\ b_3, \ldots\ldots\ldots\ldots, b_q\end{matrix}\right]$

then, using the integral due to Bhise (1, p. 74)

$$t^{\lambda-1}\,G^{l-2,\,u-1}_{p-1,\,q-2}\left[\frac{1}{t}\middle|\begin{matrix} a_2, \ldots\ldots\ldots\ldots, a_p \\ b_3, \ldots\ldots\ldots\ldots, b_q\end{matrix}\right] \underset{m}{\overset{k}{=}} \frac{1}{2^{\lambda}\quad s^{\lambda-1}\;\Gamma\,(\frac{1}{2}-k\pm m)}\times$$

$$\times\, G^{l,\,u}_{p,q}\left[2s\,\middle|\begin{matrix}\lambda+k+\frac{3}{4},\, a_2, \ldots\ldots\ldots\ldots, a_p \\ \lambda+m+\frac{1}{4},\, \lambda-m+\frac{1}{4},\, b_3, \ldots\ldots\ldots\ldots, b_q\end{matrix}\right]$$

$$= \phi\,(s\,;\,k,\,m\,;\,\lambda)$$

provided $1 \leqslant u \leqslant p,\ 2 \leqslant l \leqslant q,\ p+q+3 < 2\,(l+u)$,

$|\arg s| < (l+u-\frac{1}{2}\,p-\frac{1}{2}\,q-3/2)\,\pi$; R $(\frac{1}{2}-k\pm m) > 0$;
R $(a_i\pm m-\lambda) < 5/4$, $i=2, \ldots\ldots\ldots, u$; R $(b_j-k-\lambda+\frac{1}{4}) < 0$,
$j=3, \ldots\ldots\ldots, l$; R $(s) > 0$.

Using this in the theorems 3 and 4 and replacing $\lambda+k+\frac{3}{4}$ by a_1, $\lambda+m+\frac{1}{4}$ by b_1, $\lambda-m+\frac{1}{4}$ by b_2 and $2s$ by x, we get

$$(5.1)\quad \sum_{r=0}^{n} \frac{(-)^r}{r\,!\,(n-r)\,!\,\Gamma\,(1-a_1+b_1+r)}\, G^{l,\,u}_{p,\,q}\left[x\,\middle|\begin{matrix} a_1, \ldots\ldots\ldots\ldots, a_p \\ b_1+r,\, b_2, \ldots\ldots\ldots, b_q\end{matrix}\right]$$

$$= \frac{1}{n\,!\,\Gamma\,(1-a_1+b_1+n)}\, G^{l,\,u}_{p,q}\left[x\,\middle|\begin{matrix} a_1-n,\, a_2, \ldots\ldots\ldots\ldots, a_p \\ b_1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, b_q\end{matrix}\right]$$

and

$$(5.2)\quad \sum_{r=0}^{n} \frac{(-)^{n+r}}{r\,!\,(n-r)\,!\,\Gamma\,(a_1-b_2+r)\,\Gamma\,(1-a_1+b_2-r)}$$

$$G^{l,\,u}_{p,\,q}\left[x\,\middle|\begin{matrix} a_1+r,\, a_2, \ldots\ldots\ldots\ldots, a_p \\ b_1+r,\, b_2, \ldots\ldots\ldots\ldots, b_q\end{matrix}\right]$$

$$= \frac{\Gamma(1-a_1+b_1)}{n\,!\,\Gamma(1-a_1+b_1-n)\,\Gamma(a_1-b_2+n)\,\Gamma(1-a_1+b_2-n)} \times$$

$$\times\ G^{l,\,u}_{p,\,q}\left[x \,\middle|\, \begin{matrix} a_1+n,\ a_2,\ \ldots\ldots\ldots\ldots,\ a_p \\ b_1,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\ b_q \end{matrix}\right]$$

provided $1 \leqslant u \leqslant p,\ 2 \leqslant l \leqslant q,\ p+q < 2\,(l+u)$,
$R\,(1-a_1+b_1) > 0$ and $R\,(1-a_1+b_2) > 0$.

With $n=1$, (5.1) and with $n=1$, a replaced by $a-1$, (5.2) yield a known result (10, p. 209 (11)).

6. THEOREM 5.

If

$$L\,(s\,;\,l,\,n\,;\,\lambda) \frac{\tfrac{1}{2}+\tfrac{1}{2}\,l+n}{\pm\tfrac{1}{2}\,l}\ t^{\lambda}\,f\,(t)$$

then

$$\sum_{r=0}^{n} \frac{(1+a+b+2\,r)\,(1+a\,;\,r)\,(1+b+r\,;\,n-r)}{r\,!\,(n-r)\,!\,(1+a+b+r\,;\,n+1)} L\,(s\,;\,1+a+b+2\,r,\,n-r\,;\,\lambda)$$

$$= \frac{(2s)^{\frac{1}{2}\,(a+1)}}{n\,!} L\,(s\,;\,b,\,n\,;\,\lambda+\tfrac{1}{2}\,a+\tfrac{1}{2})$$

provided $R\,(\mu+\lambda+\tfrac{1}{2}\,a+\tfrac{1}{2}b+7/4)>0$ where $f\,(t)=0\,(t^{\mu})$ for small values of t, $f\,(t)$ is continuous for $t \geqslant t_0 > 0$, $t^{\lambda+\frac{1}{2}\,a+\frac{1}{2}\,b+7/4}\ e^{-s_0\,t}\ L^{b}_{n}\,(2s_0\,t)\,f\,(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for $R\,(s) \geqslant s_0 > 0$ and the series converges.

Proof :—We have

$$\frac{(-)^{n-r}}{(n-r)\,!} L\,(s\,;\,1+a+b+2r,\,n-r\,;\,\lambda) =$$

$$s\int_0^{\infty} (2st)^{\frac{1}{2}\,a+\frac{1}{2}\,b+r+\frac{3}{4}}\ e^{-st}\ L^{1+a+b+2r}_{n-r}\,(2st)\ t^{\lambda}\,f\,(t)\,dt.$$

Multiplying both the sides by $\dfrac{(-)^r (1+a+b+2r)(1+a\,;\,r)(1+b+r\,;\,n-r)}{r\,!\;(1+a+b+r\,;\,n+1)}$

summing for r from $r=0$ to $r=n$ and using (6, p. 38)

$$L_n^b(x) = \sum_{r=0}^{n} \frac{(-)^r (1+a+b+2r)(1+a\,;\,r)(1+b+r\,;\,n-r)}{r\,!\,(1+a+b+r\,;\,n+1)}\, x^r\, L_{n-r}^{1+a+b+2r}(x)$$

we get

$$(-)^n \sum_{r=0}^{n} \frac{(1+a+b+2r)(1+a\,;\,r)(1+b+r\,;\,n-r)}{r\,!\,(n-r)\,!\,(1+a+b+r\,;\,n+1)}\, L(s\,;\,1+a+b+2r,\,n-r\,;\,\lambda)$$

$$= s \int_0^\infty (2st)^{\frac{1}{2}a+\frac{1}{2}b+\frac{3}{4}}\, e^{-st}\, L_n^b(2st)\; t^\lambda f(t)\,dt$$

$$= (2st)^{\frac{1}{2}(a+1)}\, \frac{(-)^n}{n\,!}\, L(s\,;\,b,\,n\,;\,\lambda+\tfrac{1}{2}a+\tfrac{1}{2})\,;$$

hence the result.

The change of order of integration and summation can be justified as in Theorem 1.

6.1. Example :—

Let $f(t) = e^{-qt}$, then (2.1), on putting $k=\frac{1}{2}+\frac{1}{2}l+n$ and $m=\pm\frac{1}{2}l$ yields

$$(6.1)\qquad t^\lambda\; e^{-qt}\; \frac{\frac{1}{2}+\frac{1}{2}l+n}{\pm\frac{1}{2}l}\; \frac{\Gamma(\lambda\pm\frac{1}{2}l+5/4)}{2\,(2s)^\lambda\;\Gamma(\lambda-\frac{1}{2}l-n+5/4)}\times$$

$$\times\; {}_2F_1(\lambda+\tfrac{1}{2}l+5/4,\,\lambda-\tfrac{1}{2}l+5/4\,;\,\lambda-\tfrac{1}{2}l-n+5/4\,;\,\tfrac{1}{2}(1-q/s))$$
$$= L(s\,;\,l,\,n\,;\,\lambda)$$

$R(\lambda\pm\frac{1}{2}l+5/4)>0$, $R(s)>0$ and $|s|>|q|$.

Using this in the theorem 5, and replacing $1+a+b$ by α, $1+a$ by β, $\lambda+\frac{1}{2}a+\frac{1}{2}b+7/4$ by γ and $\frac{1}{2}(1-q/s)$ by z, we get

$$\sum_{r=0}^{n} \frac{(\alpha+2r)\,(-n\,;\,r)\,(\beta\,;\,r)\,(1+\alpha-\beta+r;\,n-r)\,(\gamma\,;\,r)}{r\,!\,(\alpha+r\,;\,n+1)\,(1+\alpha-\gamma\,;\,r)}\ {}_2F_1\left[\begin{matrix}\alpha+r,\ \gamma-\alpha-r\,; \\ \gamma-\alpha-n\end{matrix}\ z\right]$$

$$= \frac{(1+\alpha-\beta-\gamma\,;\,n)}{(1+\alpha-\gamma\,;\,n)}\ {}_2F_1\,(\gamma,\ \beta+\gamma-\alpha\,;\ \beta+\gamma-\alpha-n\,;\ z)$$

$R\,(\gamma) > 0$, $R\,(\beta+\gamma-\alpha) > 0$, $R\,(\gamma-\alpha-n) > 0$ and $|\,z\,| < 1$.

If we put $z=0$ in this result, it yields a special case of Dougall's second theorem (13, p. 372)

$${}_5F_4\left[\begin{matrix}\alpha,\ 1+\tfrac{1}{2}\,\alpha,\ \beta,\ \gamma,\ -n\ ;\ 1 \\ \tfrac{1}{2}\,\alpha,\ \alpha-\beta+1,\ a-\gamma+1,\ \alpha+n+1\end{matrix}\right] = \frac{(\alpha+1\,;\,n)\,(\alpha-\beta-\gamma+1\,;\,n)}{(a-\beta+1\,;\,n)\,(a-\gamma+1\,;\,n)}$$

$R\,(\gamma) > 0$, $R\,(\beta+\gamma-\alpha) > 0$ and $R\,(\gamma-\alpha-n) > 0$.

7. THEOREM 6

If

$$= L\,(s\,;\,a,\,r\,;\,\lambda)\ \frac{\tfrac{1}{2}+\tfrac{1}{2}\,a+r}{\pm\tfrac{1}{2}\,a}\ \ t^{\lambda}\,f\,(t)$$

and

$$\psi\,(s\,;\,\lambda) \doteqdot t^{\lambda}\,f\,(t)$$

then

$$\sum_{r=0}^{n} \frac{1}{r\,!\,(n-r)\,!\,(1+a\,;\,r)}\ L\,(s\,;\,a,\ r\,;\,\lambda)$$

$$= \frac{(2s)^{n+\frac{1}{2}\,a+\frac{1}{4}}}{n\,!\,(1+a\,;\,n)}\,\psi\,(s\,;\,\lambda+n+\tfrac{1}{2}\,a+\tfrac{3}{4})$$

provided $R\,(\lambda+\mu+\tfrac{1}{2}\,a+5/4) > 0$ where $f\,(t)=0\,(t^{\mu})$ for small values of t, (t) is

continuous for $t \geqslant t_0 > 0$, $t^{\lambda+n+\frac{1}{2}a+5/4}\, e^{-s_0 t} f(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for $R(s) \geqslant s_0 > 0$ and the series converges

Proof :—We have

$$\frac{1}{r\,!}\, L(s\,;\, a,\, r\,;\, \lambda) = (-)^r s \int_0^\infty (2st)^{\frac{1}{2}a+\frac{1}{4}}\, e^{-st}\, L_r^a (2st)\, t^\lambda f(t)\, dt.$$

Multiplying both the sides by $\dfrac{1}{(n-r)\,!\,(1+a\,;\,a)}$, summing for r from $r=0$ to $r=n$

and using (13, p. 207)

$$\sum_{r=0}^{n} \frac{(-)^r\, L_r^a(x)}{(n-r)\,!\,(1+a\,;\,r)} = \frac{x^n}{n\,(1+a\,!\,n)},$$

we get the result.

7.1. Example :—

Using (6.1) and (2.2) in the theorem 6, and replacing $\lambda+\frac{1}{2}a+5/4$ by α, $\lambda-\frac{1}{2}a+5/4$ by β, and $\frac{1}{2}(1-q/s)$ by z, we get

$$\sum_{r=0}^{n} \frac{(-)^r (1-\beta\,;\,r)}{r\,!\,(n-r)\,!\,(\alpha-\beta+1\,;\,r)}\, {}_2F_1(\alpha, \beta\,;\,\beta-r\,;\,z)$$

$$= \frac{(\alpha\,;\,n)}{n\,!\,(\alpha-\beta+1\,;\,n)}\,(-z)^{-(\alpha+n)}$$

$R(\alpha) > 0$, $R(\beta) > 0$ and $|z|\,1$.

8, THEOREM 7.

If

$$L(s\,;\, l,\, n;\, \lambda)\, \frac{\frac{1}{2}+\frac{1}{2}l+n}{\pm\frac{1}{2}l}\, x^\lambda f(x)$$

and

$$\psi(s\,;\lambda\,;t) \doteq \left(\frac{x}{1+2t}\right)^{\lambda} f\left(\frac{x}{1+2t}\right)$$

then

$$\sum_{r=0}^{\infty} \frac{(-)^r (2s)^{\frac{1}{2}r}\, t^r}{r\,!} \mathrm{L}\,(s\,;a-r,\, r\,;\lambda+\tfrac{1}{2}r)$$

$$= \frac{(2s)^{\frac{1}{2}a+\frac{1}{4}}\,(1+t)^a}{(1+2t)}\;\psi\,(s\,;\lambda+\tfrac{1}{2}a+\tfrac{1}{4}\,;t)$$

provided $\mathrm{R}\,(\mu+\lambda+\frac{1}{2}a+5/4) > 0$, where $f(x)=0\,(x^{\mu})$ for small values of x, $f(x)$ is continuous, $\mathrm{R}\,(s) \geqslant s_0 > 0$ and $|\,t\,| > \frac{1}{2}$.

Proof :—We have

$$\frac{(-)^r}{r\,!}\,\mathrm{L}\,(s\,;a-r,\, r\,;\lambda+\tfrac{1}{2}r)$$

$$= s\int_0^{\infty} (2sx)^{\frac{1}{2}a-\frac{1}{2}r+\frac{1}{4}}\; e^{-sx}\;\; \mathrm{L}_r^{a-r}\,(2sx)\;\; x^{\lambda+\frac{1}{2}r}\, f(x)\, dx.$$

Multiplying both the es by $(2s)^{\frac{1}{2}r}\, t^r$, summing for r from $r=0$ to $r=\infty$, and using (7, p. 222),

$$\sum_{r=0}^{\infty} \mathrm{L}_r^{a-r}\,(z)\, t^r = (1+t)^a\, e^{-zt}\text{, we get}$$

$$\sum_{r=0}^{\infty} \frac{(-)^r (2s)^{\frac{1}{2}r}\, t^r}{r\,!}\,\mathrm{L}\,(s\,;a-r,\, r\,;\lambda+\tfrac{1}{2}r)$$

$$= s\,(1+t)^{a}\,(2s)^{\frac{1}{2}a+\frac{1}{4}} \int_{0}^{\infty} e^{sx(1+2t)}\, x^{\lambda+\frac{1}{2}a+\frac{1}{4}}\, f(x)\, dx$$

$$= \frac{(2s)^{\frac{1}{2}a+\frac{1}{4}}(1+t)^{a}}{1+2t}\, s \int_{0}^{\infty} e^{-sx} \left(\frac{x}{1+2t}\right)^{\lambda+\frac{1}{2}a+\frac{1}{4}} f\left(\frac{x}{1+2t}\right) dx$$

on replacing x by $x/(1+2t)$; hence the result.

8.1. Example :—

Using (6.1) and (2.2) in theorem 7, and replacing $\lambda+\frac{1}{2}a+5/4$ by α, $\lambda-\frac{1}{2}a+5/4$ by β, and $\frac{1}{2}(1-q/s)$ by z, we get

$$\sum_{r=0}^{\infty} \frac{(-)^{r}\, t^{r}\, (\beta\,;r)}{r\,!}\; {}_2F_1\,(\alpha,\ \beta+r\,;\ \beta\,;\ z)$$

$$= (1+t)^{\alpha-\beta}\,(1+t-z)^{-\alpha}$$

$R(\alpha) > 0$, $R(\beta) > 0$ and $|z| < 1$.

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ON H-FUNCTION OF FOX

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ABSTRACT

Fox has defined a function.

$$H(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{m=1}^{q}\Gamma(b_m+c_m s)\prod_{n=1}^{p}\Gamma(a_n-e_n e)}{\prod_{m=1}^{q}\Gamma(b_m+c_m-c_m s)\prod_{n=1}^{p}\Gamma(a_n-e_n+e_n s)}\,x^{-s}\,ds$$

which is more general than the Meijer's G. Function. He has proved that it is a symmetric Fourier kernel and has pointed out that it is a most general function that needs further study.

With the help of L^2—theory of transform given by waston it can be shown that the kernel plays the role of a transform. Let us therefore define

$$f(p)=\int_0^\infty H(px)\,f(x)\,dx$$

as the Fox-transform of $f(x)$ provided the integral on the right converges.

In the present paper the author obtains two theorems giving some formulae of Poisson type in case of this transform.

1. *Introduction* : A more general function than the G-Function of Meijer has been defined by Fox (2) by a Mellin-Barne's type integral.

$$(1.1)\quad H(x)=\frac{1}{2\pi i}\int_T P(s)\,x^{-s}\,ds$$

where

$$(1.2)\quad P(s)=\frac{\prod_{m=1}^{q}\Gamma(b_m+c_m s)\prod_{n=1}^{p}\Gamma(a_n-e_n s)}{\prod_{m=1}^{q}(b_m+c_m-c_m s)\prod_{n=1}^{p}\Gamma(a_n-e_n+e_n s)}$$

and

(i) $c_m > 0,\ m = 1, 2, \ldots\ldots\ldots q.$
$c_n > 0.\ n = 1, 2, \ldots\ldots\ldots p.$

(ii) all the poles of the integrand are simple,

(iii) The contour T is a straight line paralled to the imaginary axis in the $s\,(= \sigma + it)$-plane and the poles of $\Gamma\,(b_m + c_m s)$ lie on the left of T while those of $\Gamma\,(a_n - e_n s)$ lie on the right of T

(iv) $D = 2 \left[\sum_{m=1}^{q} c_m - \sum_{n=1}^{p} e_n \right] > 0$

It may be noted that if $\sigma < \frac{1}{2}$ the integral (1.1) is uniformly convergent with respect to x.

Fox has further proved that the function $H(x)$ is a symmetric fourier kernel and that it is a most general function that needs further study.

With the help of L^2 — theory known as the theory of K-Transform given by watson (5) it can be established that the kernel defined by (1.1) plays the role of a transform. Let us therefore define.

$$I(p) = \int_0^\infty H(px)\, f(x)\, dx$$

as the Fox Transform of $f(x)$ provided the integral on the right is convergent.

In this paper we obtain two theorems in which some formulae of poisson type (3, pp. 60-66) have been derived in case of this transform.

The following result in connection with Mellin Transform shall be required in our discussion (3. p. 95).

If $\phi(s)$ and $\psi(s)$ be the Mellin Transforms of $f(x)$ and $g(x)$ respectively then

$$(1.4) \qquad \int_0^\infty f(ax)\, g(bx)\, dx = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \phi(1-s)\, \psi(s)\, a^{s-1}\, b^{-s}\, dx$$

provided that $x^\sigma f(x)$ and $x^{1-\sigma} g(x)$ belong to L^2

2. *Theorem. 1.* If $x^\sigma f(x)$ belongs to L^2 and if $F(x)$ is the Fox-Transform of $f(x)$ then

$$(2.1)\quad \sum_{r=1}^{\infty} (-1)^{r+1} F(\overline{2r-1}\, x) = \sqrt{\pi} \sum_{r=1}^{\infty} (-1)^{r+1} \int_0^{\infty} f(\overline{2r-1}\, \pi t)\, Q(xt)\, dt$$

where

$$(2.2)\quad Q(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{2^{2s-1}} \cdot \frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)} P(s)\, x^{-s}\, ds.$$

Provided that

(i) the series are convergent

(ii) $\Gamma(b_m + c_m \sigma) > 0,\ m = 1, 2, \ldots\ldots q$

$\Gamma(a_n - e_n \sigma) > 0,\ n = 1, 2, \ldots\ldots p$

and $\sigma < -\frac{1}{2}$

Proof: Consider the integral

$$I = \sqrt{\pi} \sum_{r=1}^{\infty} (-1)^{r+1} \int_0^{\infty} f(\overline{2r-1}\, \pi t)\, Q(xt)\, dt$$

$$= \sqrt{\pi} \sum_{r=1}^{\infty} (-1)^{r+1} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{2^{2s-1}} \cdot \frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)} P(s)\, \phi(1-s)\, (\overline{2r-1}\, \pi)^{s-1}\, x^{-s} ds$$

$$(s = \sigma + it,\ c < -\tfrac{1}{2})$$

(using (1.4)

Now changing the order of summation and integration which is justified by uniform convergence, we obtain,

$$I = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\pi^{s-\frac{1}{2}}}{2^{2s-1}} \cdot \frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)} P(s)\, \Phi(1-s)\, L(1-s)\, x^{-s}\, ds$$

where

$$(2.3)\quad L(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \ldots\ldots \qquad R(s) > 1$$

Now L (s) is an integral function of s satisfying the functional relation (4, p. 298)

$$(2.4)\quad L(1-s) = \left(\frac{2}{\pi}\right)^{s-\frac{1}{2}} \sin(\tfrac{1}{2} s \pi)\, \Gamma(s)\, L(s)$$

Then using duplication formula (1, p. 5)

$$(2.5)\quad \Gamma(s) = \pi^{-\frac{1}{2}}\, 2^{s-1}\, \Gamma(\tfrac{1}{2}s)\, \Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)$$

we have

$$I = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \phi(1-s)\, P(s)\, L(s)\, x^{-s}\, dx$$

$$= \sum_{r=1}^{\infty} (-1)^{r+1} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \phi(1-s)\, P(s) \left\{ \overline{2r-1}\, x \right\}^{-s} ds.$$

$$= \sum_{r=1}^{\infty} (-1)^{r+1} \int_0^{\infty} H(\overline{2r-1}\, xy)\, f(y)\, dy$$

$$= \sum_{r=1}^{\infty} (-1)^{r+1}\, F(\overline{2r-1}\, x)$$

Hence the result,

3. *Theorem II.* Under the same conditions as in theorem **I**,

$$(3.1)\quad \sum_{r=1}^{\infty} F(rx) = R + (4\pi)^{\frac{1}{2}} \sum_{r=1}^{\infty} \int_0^{\infty} f(4\pi r t)\, Q(xt)\, dt$$

where

$$(3.2)\quad Q(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{2^{2s-1}} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\, P(s)\, x^{-s}\, ds$$

and

$$(3.3)\quad R = \frac{1}{x}\, \frac{\prod_{m=1}^{q} \Gamma(b_m+c_m) \prod_{n=1}^{p} \Gamma(a_n-e_n)}{\prod_{m=1}^{q} \Gamma(b_m) \prod_{n=1}^{p} \Gamma(a_n)}\, \phi(0),$$

if $\phi(s)$ has no pole in $(c, 1+b)$; $c < 0 < b$.

But if $\phi(s)$ has a simple pole at $s=0$ we have

(3.4) $\quad R = \underset{s \to 1}{Lt} \frac{d}{ds} \left[(s-1)^2 \ \phi(1-s) \ \xi(s) \ P(s) \ x^{-s} \right]$

$\xi(s)$ being the Riemann Zeta-function defined by

(3.5) $\quad \xi(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots\ldots \qquad R(s) > 1$

Proof :—Consider

$$I = (4\pi)^{\frac{1}{2}} \sum_{r=1}^{\infty} \int_0^{\infty} f(4\pi rt)\, Q(xt)\, dt$$

$$= (4\pi)^{\frac{1}{2}} \sum_{r=1}^{\infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{2^{2s-1}} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} P(s)\, \phi(1-s)\, (4\pi r)^{s-1} x^{-s}\, ds$$

$(s = \sigma + it\, ;\, c < -\frac{1}{2}) \qquad$ [by (1.4).]

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (4\pi)^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2^{2s-1}\,\Gamma(\frac{1}{2}s)} P(s)\, \phi(1-s) \sum_{r=1}^{\infty} r^{s-1} x^{-s}\, ds.$$

the change of order of summation and integration being justifiable as before.

Now using (3.5) and noting that $\xi(s)$ satisfies the functional relation (6. p. 2 9)

(3.6) $\quad \pi^s \xi(1-s) = 2^{1-s} \cos(\frac{1}{2} s\pi)\, \Gamma(s)\, \xi(s)$

we obtain

$$I = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(1-s)\, P(s)\, \xi(s)\, x^{-s}\, ds$$

Moving the line of integration from $\sigma = c$ to $\sigma = 1 + b$ where $b > 0$ and $c < 0 < b$ and observing that $\phi(1-s)$ has a simple pole at $s = 1$ with residue $= 1$ and (3, p-63).

$$\Phi(s) = \frac{f(0)}{s} + \int_0^1 \left\{ f(x) - f(0) \right\} x^{s-1}\, dx + \int_1^{\infty} x^{s-1} f(x)\, dx$$

has in general a simple pole at $s=0$ with residue $f(0)$, we have

$$(3.7) \quad I + R = \frac{1}{2\pi i} \int_{1+b-i\infty}^{1+b+i\infty} \phi(1-s)\, P(s)\, \xi(s)\, x^{-s}\, ds.$$

where R is the residue of the integrand at the pole $s = 1$. There are two cases.

Case I. When $\phi(s)$ has no pole in (c, 1+b) ; $c < o < b$. Evidently in this case.

$$R = P(1)\, \Phi(o)\, x^{-1}$$

$$= \frac{1}{x} \frac{\prod_{m=1}^{q} \Gamma(b_m + e_m) \prod_{n=1}^{p} \Gamma(a_n - e_n)}{\prod_{m=1}^{q} \Gamma(b_m) \prod_{n=1}^{p} \Gamma(a_n)} \Phi(o)$$

Case II. When $\phi(s)$ has a simple pole at $s=o$

Here

$$R = \underset{s \to 1}{Lt} \frac{d}{ds} \left[(s-1)^2\, \phi(1-s)\, P(s)\, \xi(s)\, x^{-s} \right]$$

Hence.

$$I + R = \frac{1}{2\pi i} \int_{1+b+i\infty}^{1+b+i\infty} \Phi(1-s)\, P(s) \sum_{r=1}^{\infty} (rx)^{-s}\, ds$$

$$= \frac{1}{2\pi i} \sum_{r=1}^{\infty} \int_{1+b-i\infty}^{1+b+i\infty} \phi(1-s)\, P(s)\, (rx)^{-s}\, ds$$

$$= \sum_{r=1}^{\infty} \int_{0}^{\infty} H(r\, xy)\, f(y)\, dy, \qquad \text{Using (1.4)}$$

$$= \sum_{r=1}^{\infty} F(rx)$$

and the proof is complete.

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ON POLYNOMIALS RELATED TO THE ULTRASPHERICAL POLYNOMIALS

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ABSTRACT

In this paper few generating functions, hypergeometric forms, and some properties of the polynomials given by L. Carlitz have been obtained. The results are believed to be new.

1. Carlitz [1] has introduced a class of polynomials related to the ultraspherical polynomials defined as follows :

$$\sum_{r=0}^{n} A_r^{(\lambda)}(x)\, C_{n-r}^{(\lambda+r)}(x) = 0,\ n \geqslant 1$$

$$A_0^{(\lambda)}(x) = 1.$$

In it he also obtained

$$(1.1)\quad A_n^{(\lambda)}(x) = (-1)^n \sum_{r=0}^{[n/2]} \frac{(\lambda)_n\,(2x)^{n-2r}}{r\,!\,(n-2r)\,!\,(1+\lambda)_r}$$

$$(1.2)\quad A_n^{(\lambda)}(x) = (-1)^n A_n^{(\lambda)}(x)$$

and

$$(1.3)\quad A_n^{(\lambda)}(x) = \frac{\lambda}{\lambda+n}\, C_n^{-\lambda-n}(x)$$

The Rodrigues formula for (1.1) is given by [1, p. 130] with a slight misprint. The correct formula is

$$(1.4)\quad A_n^{(\lambda)}(x) = \frac{(-2)^n(\lambda)_n}{(2\lambda+1)_n\, n\,!}\,(1-x^2)^{\frac{1}{2}+n+\lambda}\,.D^n\,(1-x^2)^{-\lambda-\frac{1}{2}}$$

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2. GENERATING FUNCTIONS.

From (1.4) we easily obtain the generating function

$$(2.1)\quad (1-x^2)^{\lambda+\frac{1}{2}}\left[1-\{x+(1-x^2)\,t\}^2\right]^{-\lambda-\frac{1}{2}}$$

$$=\sum_{n=o}^{\infty}\frac{(2\lambda+1)_n\,t^n}{(-2)^n(\lambda)_n}A_n^{(\lambda)}(x).\qquad |t|<1$$

From [1, p/28] we have

$$(z/t)^{-\lambda}=\sum_{n=o}^{\infty}z^n A_n^{(\lambda)}(x)$$

where $t=\frac{1}{2z}\left[1+2xz-\{(1+2xz)^2-4z^2\}^{\frac{1}{2}}\right]$

Therefore

$$\sum_{n=o}^{\infty}\sum_{m=o}^{\infty}\frac{(n+m)!}{n!\,m!}u^n v\quad A_{n+m}^{(\lambda)}(x)$$

$$=\sum_{n=o}^{\infty}\sum_{m=o}^{n}\frac{n!\,u^{n-m}v^m}{(n-m)!\,m!}A_n^{(\lambda)}(x)$$

$$=\sum_{n=o}^{\infty}(u+v)^n A_n^{(\lambda)}(x)$$

$$(2.2)\qquad =\left(\frac{u+v}{t}\right)^{-\lambda}$$

where t is given by

$$\frac{u+v}{t}=\frac{1}{1-2xt+t^2}$$

Also using the relation [1, p. 131]

$$\sum_{n=o}^{\infty} \frac{A_n^{(\lambda)}(x)\, t^n}{(\lambda)_n} = \Gamma(\lambda+1)\, t^{-\lambda}\, e^{-2xt}\, I_\lambda(2t)$$

$$\sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{(n+m)!}{n!\, m!} \cdot \frac{A_{n+m}^{(\lambda)}(x)\, u^n v^m}{(\lambda)_{n+m}}$$

$$= \sum_{n=o}^{\infty} \frac{(u+v)^n A_n^{(\lambda)}(x)}{(\lambda)_n}$$

$$= \Gamma(\lambda+1)\,(u+v)^{-\lambda}\, e^{-2x(u+v)} \times I_\lambda\{2(u+o)\}$$

From (1.4) we have

$$A_n^{(\lambda)}(x) = \frac{(-2)^n (\lambda)_n}{(2\lambda+1)_n\, n!} (1-x^2)^{\frac{1}{2}+n+\lambda} \cdot D^n [(1-x)^{-\lambda-\frac{1}{2}} (1+x)^{-\lambda-\frac{1}{2}}]$$

Using Leibnitz's theorem, we get

$$A_n^{(\lambda)}(x) = \frac{(-2)^n (\lambda)_n}{(2\lambda+1)_n\, n!} (1-x^2)^{\frac{1}{2}+n+\lambda}$$

$$\times \sum_{k=o}^{n} \frac{n!\, (-1)^{n-k} (1-x)^{-\lambda-\frac{1}{2}-k}}{(n-k)!\, k!}$$

$$\times (\lambda+\tfrac{1}{2})_k\, (\lambda+\tfrac{1}{2})_{n-k}\, (1+x)^{-\frac{1}{2}-n-\lambda+k}$$

$$(2.4) \quad = \frac{(-2)^n (\lambda)_n}{(2\lambda+1)_n\, n!} \sum_{k=o}^{n} \frac{n!\, (-1)^{n-k}}{(n-k)!\, k!} (1-x)^{n-k} (\lambda+\tfrac{1}{2})_k\, (\lambda+\tfrac{1}{2})_{n-k}\, (1+x)^k$$

Hence

$$\sum_{n=0}^{\infty} \frac{(2\lambda+1)_n\, t^n}{(-2^n)\,(\lambda)^n}\; A_n^{(\lambda)}(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\lambda+\frac{1}{2})_k\,(x+1)^k\, t^n}{k\,!\,(n-k)\,!}\,(x-1)^{n-k}\,(\lambda+\tfrac{1}{2})_{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda+\frac{1}{2})_n\,(x+1)^n\, t^n}{n\,!} \sum_{n=0}^{\infty} \frac{(\lambda+1)_n\,(x-1)^n\, t^n}{n\,!}$$

$$(2.5) \qquad = {}_1F_0\,[\lambda+\tfrac{1}{2},\, -,\, (x+1)\, t].\ {}_1F_0\,[\lambda+\tfrac{1}{2},\, -,\, (x-1)\, t]$$

giving another generating function.

3. HYPER GEOMETRIC FORMS OF $A_n^{(\lambda)}(x)$

From (1.1) we obtain

$$(3.1) \quad A_n^{(\lambda)}(x) = \frac{(-1)^n\,(\lambda)_n\,(2x)^n}{n\,!}\; {}_2F_1\left[\, -\tfrac{1}{2}n,\, -\tfrac{1}{2}n+\tfrac{1}{2},\, 1+\lambda,\, 1/x^2 \,\right]$$

From (2.4) we obtain

$$(3.2) \quad A_n^{(\lambda)}(x) = \frac{(-2)^n\,(\lambda)_n\,(\lambda+\frac{1}{2})_n\,(x-1)^n}{(2\lambda+1)_n\, n\,!}\,.\; {}_2F_1\left[\begin{matrix}-n,\ \lambda+\frac{1}{2}\\ \frac{1}{2}-n-\lambda\end{matrix},\ \left(\frac{x+1}{x-1}\right)\right]$$

Reversing the order of summation of (3.2) we get

$$(3.3) \quad A_n^{(\lambda)}(x) = \frac{(-2)^n\,(\lambda)_n\,(\lambda+\frac{1}{2})_n\,(x+1)^n}{(2\lambda+1)_n.\, n\,!} \times {}_2F_1\left[\begin{matrix}-n,\ \lambda+\frac{1}{2}\\ \frac{1}{2}-n-\lambda\end{matrix},\ \frac{x-1}{x+1}\right]$$

In (2.4) expanding $(1-x)^{n-k}$ in powers of $\left(\frac{1+x}{2}\right)$, changing the order of summation and summing the first series we obtain

$$(3.4) \quad A_n^{(\lambda)}(x) = \frac{2^{2n}\,(\lambda)_n\,(\lambda+\frac{1}{2})_n}{(2\lambda+1)_n\, n\,!}\,.\; {}_2F_1\left[\begin{matrix}-n,\ -2\lambda-n\\ -\lambda-n+\frac{1}{2}\end{matrix},\ \frac{1+x}{2}\right]$$

Using (1.2) in (3.4) we get

$$(3.5) \quad A_n^{(\lambda)}(x) - \frac{(-1)^n 2^{2n}\,(\lambda)_n\,(\lambda+\frac{1}{2})_n}{(2\lambda+1)_n\, n\,!}\,.\; {}_2F_1\left[\begin{matrix}-n,\ -2\lambda-n,\\ -\lambda-n+\frac{1}{2},\end{matrix}\ \frac{1-x}{2}\right]$$

4. Rainville [2] has obtained a series for Legendre polynomials. A similar expression for $A_n^{(\lambda)}(x)$ has been obtained here

From [1, p. 131] we have

$$\sum_{n=0}^{\infty} \frac{A_n^{(\lambda)}(x)\, t^n}{(\lambda)_n} = \Gamma(\lambda+1)\, t^{-\lambda}\ \lambda\,(2t) e^{-2xt}$$

Therefore

$$\sum_{n=0}^{\infty} \frac{A_n^{(\lambda)}(\tan\delta)\, t^n}{(\lambda)_n} = \Gamma(\lambda+1)\, t^{-\lambda}\ I\lambda\ (2\ t) e^{-2\tan\delta .\, t}$$

and

$$\sum_{n=0}^{\infty} \frac{A_n^{(\lambda)}(\tan\gamma)\, t^n}{(\lambda)_n} = \Gamma(\lambda+1)\, t^{-\lambda}\ I_\lambda\ (2\ t)\ e^{-2\tan\gamma .\, t}$$

giving

$$(4.1)\qquad \sum_{n=0}^{\infty} \frac{A_n^{(\lambda)}(\tan\delta)\, t^n}{(\lambda)_n} = e^{2t(\tan\gamma - \tan\delta)} \sum_{n=0}^{\infty} \frac{A_n^{(\lambda)}(\tan\gamma)\, t^n}{(\lambda)_n}$$

Equating the coefiecients of t^n of both the sides of (4.1) we obtain

$$(4.2)\qquad A_n^{(\lambda)}(\tan\delta) = \sum_{r=0}^{n} \frac{(2)^r}{r!} \left\{ \frac{\sin(\gamma-\delta)}{\cos\gamma\ \cos\delta} \right\}^r A_{n-r}^{(\lambda)}(\tan\gamma) \times (-1)^r (1-\lambda-n)_r$$

Also from [1, p. 131] we have

$$\sum_{r=0}^{\infty} \frac{A_r^{(\lambda)}(x)\, t^r}{(\lambda)_r} = e^{-2xt}\ {}_0F_1\,[1+\lambda,\ t^2]$$

which can also be written as

$$(4.3)\qquad e^{2xt} \sum_{r=0}^{\infty} \frac{A_r^{(\lambda)}(x)\, t^r}{(\lambda)_r} = {}_0F_1\,[1+\lambda,\ t^2]$$

Equating the coefficients of t^n of both the sides of (4.3) we get

$$\sum_{r=o}^{n} \frac{A_r^{(\lambda)}(x)\cdot(2x)^{n-r}}{(\lambda)_r\,(n-r)!} = 0 \text{ If } n \text{ is odd}$$

$$= \frac{1}{(1+\lambda)_{n/2}\,(n/2)!} \text{ If } n \text{ is even.}$$

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ON CERTAIN KERNEL FUNCTIONS

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ABSTRACT

In this paper, two rules, connecting different classes of self-reciprocal functions in transform.

$$g(x) = 2\gamma\ \beta^{1/2\gamma}$$

$$\int_0^\infty G_{2p,\,2q}^{q,\,p}\left[\beta^2 (xy)^{2\gamma}\left|\begin{matrix}\frac{2\gamma-1}{4\gamma}+a_1, \ldots, \frac{2\gamma-1}{4\gamma}+a_p, \frac{2\gamma-1}{4\gamma}-a_1, \ldots, \frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1, \ldots, \frac{2\gamma-1}{4\gamma}+b_q, \frac{2\gamma-1}{4\gamma}-b_1, \ldots, \frac{2\gamma-1}{4\gamma}-b_q\end{matrix}\right.\right] \times f(y)\, dy,$$

based on a symmetrical Fourier kernel, given by R. Narain, have been stated. Using these rules some new kernels have been obtained and various particular cases have been discussed.

1. A new generalisation of the Hankel transform may be introduced in the form

$$(1.1)\qquad g(x) = 2\gamma\beta^{1/2\gamma}$$

$$\int_0^\infty G_{2p,\,2q}^{q,\,p}\left[\beta^2 (xy)^{2\gamma}\left|\begin{matrix}\frac{2\gamma-1}{4\gamma}+a_1, \ldots, \frac{2\gamma-1}{4\gamma}+a_p, \frac{2\gamma-1}{4\gamma}-a_1, \ldots, \frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1, \ldots, \frac{2\gamma-1}{4\gamma}+b_q, \frac{2\gamma-1}{4\gamma}-b_1, \ldots, \frac{2\gamma-1}{4\gamma}-b_q\end{matrix}\right.\right] \times f(y)\, d\ ,$$

with the help of a symmetrical Fourier kernel, given recently by R. Narain (7, p. 951), in terms of Meijer's G-function as

$$(1.2)\qquad 2\beta\gamma\ x^{\gamma-\frac{1}{2}}\ G_{2p,\,2q}^{q,\,p}\left[\beta^2 x^{2\gamma}\left|\begin{matrix}a_1, \ldots, a_p, -a_1, \ldots, -a_p\\ b_1, \ldots, b_q, -b_1, \ldots, -b_q\end{matrix}\right.\right],$$

where β and γ are real constants.

If the functions $f(x)$ and $g(x)$ are identical, then $f(x)$ will be called as self-reciprocal in (1.1).

On setting the parameters suitably, (1.2) yields, as particular cases, various Fourier kernels.

(A) With $\beta = \frac{1}{2}$ and $\gamma = 1$ (1.2) can be reduced to another Fourier kernel (5, p. 298)

$$(1.3) \quad \sqrt{x}\; G^{q,\,p}_{2p,\,2q}\left[\frac{x^2}{4}\middle| \begin{matrix} a_1, \ldots, a_p, -a_1, \ldots, -a_p \\ b_1, \ldots, b_q, -b_1, \ldots, -b_q \end{matrix}\right].$$

(B) Taking $\gamma = 1$, $\beta = 2^{-n}$, where, n is a positive integer, and giving suitable values to the parameters, (1.2) reduces to

$$(1.4) \quad 2^{1-n}\, x^{\frac{1}{2}}\, G^{n,\,o}_{o,\,2n}\left[\frac{x^2}{2^{2n}}\middle/ \frac{\mu_1}{2}, \ldots, \frac{\mu_n}{2}, \frac{-\mu_1}{2}, \ldots, \frac{-\mu_n}{2}\right] \equiv \omega_{\mu_1, \ldots, \mu_n}(x),$$

where $\omega_{\mu_1, \ldots, \mu_n}(x)$ is the kernel, defined by Bhatnagar (2, p. 43).

(*i*) When $n = 1$, this gives the Hankel's kernel $x^{\frac{1}{2}}\, J_\nu(x)$.

(*ii*) If $n = 2$, (1.4) yields.

$$(1.5) \quad \tfrac{1}{2}\, x^{\frac{1}{2}}\, G^{2,\,o}_{o,\,4}\left[\frac{x^2}{16}\middle| \frac{\mu}{2}, \frac{\nu}{2}, \frac{-\mu}{2}, \frac{-\nu}{2}\right] \equiv \omega_{\mu,\,\nu}(x),$$

where $\omega_{\mu,\,\nu}(x)$ is the kernel, given by Watson (11, p. 308).

(C) On putting $\beta = \left(\frac{1}{2k}\right)^k$, $\gamma = k$ and adjusting the parameters in (1.2) suitably, we obtain Everitt's kernels (3, p. 271).

$$(1.6) \quad x^{\frac{1}{2}}\, J_{o,\,k}(x) \equiv (2k)^{\frac{1}{2}} \left(\frac{x}{2k}\right)^{k-\frac{1}{2}}$$

$$G^{k,0}_{0,2k}\left[\left(\frac{x}{2k}\right)^{2k}\middle|\, 0, \frac{1}{2k}, \ldots, \frac{k-1}{2k}, 0, -\frac{1}{2k}, \ldots, -\frac{k-1}{2k}\right],$$

which reduces to the kernel of the Hankel transform of order zero when $k=1$,

and

$$(1.7)\quad x^{\frac{1}{2}} J_{\frac{1}{2},k}(x) \equiv (2k)^{\frac{1}{2}}\left(\frac{x}{2k}\right)^{k-\frac{1}{2}}$$

$$G^{k,0}_{0,2k}\left[\left(\frac{x}{2k}\right)^{2k}\middle|\, \frac{1}{4k}, \frac{3}{4k}, \ldots, \frac{2k-1}{4k}, -\frac{1}{4k}, -\frac{3}{4k}, \ldots, -\frac{2k-1}{4k}\right],$$

which gives the kernel for sine transform if $k=1$.

(D) Substituting $\beta=\frac{1}{2}$, $\gamma=1$, $p=1$, $q=2$ and choosing the parameters in (1.2) suitably, we have

$$(1.8)\quad \sqrt{2}\, G^{2,1}_{2,4}\left[\frac{x^2}{4}\middle|\begin{matrix} k-m-\frac{\nu}{2}-\frac{1}{4}, & -k+m+\frac{\nu}{2}+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}+m\pm m, & -\frac{\nu}{2}+\frac{1}{4}-m\pm m\end{matrix}\right]$$

$$\equiv \frac{1}{2^{\nu}}\, x^{\nu+\frac{1}{2}}\, X_{\nu,k,m}\left(\frac{x^2}{4}\right),$$

which plays the role of a kernel in another generalisation of Hankel transform (4, p. 271).

Earlier the author (9) has investigated the condition under which a function can be self-reciprocal in (1.1) and has given some new self-reciprocal functions. The object of this paper is to state some rules connecting different classes of self-reciprocal functions in (1.1) and to use these in evaluating some new kernels. The importance of these kernels lies in the fact that some known as well as unknown kernels can be deduced from these, as particular cases, by suitable substitutions.

2. RULE 1:—If $f(x)$ is a self-reciprocal function in (1.1) and belongs to A (α, a), the function

$$(2.1)\quad g(x) = \frac{1}{x}\int_0^\infty P\left(\frac{y}{x}\right) f(y)\, dy$$

is self-reciprocal in (1.1) with a's and b's replaced by c's and d's, provided that

$$(2.2)\quad P(x) = \frac{1}{2\pi i}\times$$

$$\int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{q} \Gamma\left(\frac{2\gamma+1}{4\gamma}+d_j-\frac{s}{2\gamma}\right) \chi(s)\, x^{-s}}{\prod_{j=1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{p} \Gamma\left(\frac{2\gamma+1}{4\gamma}-c_j-\frac{s}{2\gamma}\right)}\, ds,$$

where $\chi(s)$ is regular and satisfies the condition

$$(2.3)\quad \chi(s) = \chi(1-s)$$

in the strip

$$(2.4)\quad \alpha < \sigma < (1-a)$$

and

$$(2.5)\quad \chi(s) = O\left(e^{\{(q-p)\pi/2-\alpha+\eta\}\,|t|}\right)$$

for every η and uniformly in any strip interior to (2.4) and c is any value of σ in (2.4).

In other words $P(x)$ is a kernel transforming a self-reciprocal function in (1.1) into another self-reciprocal function in the same transform with a's and b's replaced by c's and d's.

RULE 2 : If $f(x)$ is a self-reciprocal function in (1.1) and belongs to A (α, a), then the function

$$(2.6)\quad g(x) = \int_0^\infty P(xy)\, f(y)\, dy$$

is self-reciprocal in (1.1) with a's and b's replaced by c's and d's, provided that

$$(2.7)\quad P(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}$$

$$\beta^{-s/\gamma} \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{q}\Gamma\left(\frac{2\gamma-1}{4\gamma}+d_j+\frac{s}{2\gamma}\right)\chi(s)\,x^{-s}}{\prod_{j=1}^{p}\Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{p}\Gamma\left(\frac{2\gamma-1}{4\gamma}-c_j+\frac{s}{2\gamma}\right)}ds,$$

where $\chi(s)$ satisfies the conditions (2.3) and (2.5).

Since the result is symmetrical, the kernel (2.7) effects the converse transformation also.

In other words $P(x)$ is a kernel transforming a self-reciprocal function in (1.1) into another self-reciprocal function in (1.1) with a's and b's replaced by c's and d's and vice versa.

The proofs of the above rules can be developed on the same lines as in the proofs of the corresponding rules of the Hankel transform (10, pp. 268-270). It should be assumed that the function $f(x)$ is integrable in (o, ∞) and the integrals, involved, exist.

3. Consider the function

$$G^{q+r,\, q+r}_{p+q+k,\, p+q+k}\left[x^{\rho}\,\middle|\,\begin{matrix}\alpha_1, \ldots, \alpha_{p+q+k}\\ \beta_1, \ldots, \beta_{p+q+k}\end{matrix}\right],\ q+2r > p+k > r.$$

By definition (1, p. 207) this equals to

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{q}\Gamma(\beta_j-t)\prod_{j=1}^{q}\Gamma(1-\alpha_j+t)}{\prod_{j=1}^{p}\Gamma(1-\beta_{q+k+j}+t)\prod_{j=1}^{p}\Gamma(\alpha_{q+k+j}-t)}$$

$$\frac{\prod_{j=1}^{r}\Gamma(\beta_{q+j}-t)\prod_{j=1}^{r}\Gamma(1-\alpha_{q+j}+t)}{\prod_{j=r+1}^{k}\Gamma(1-\beta_{q+j}+t)\prod_{j=r+1}^{k}\Gamma(\alpha_{q+j}-t)}\,x^{\rho t}\,dt.$$

With $\rho=2\gamma$ and replacing t by $-\dfrac{s}{2\gamma}-\dfrac{2\gamma-1}{4\gamma}$ we get

$$(3.1)\quad x^{\gamma-\frac{1}{2}}\, G^{q+r,\, q+r}_{p+q+k,\, p+q+k}\left[x^{2\gamma}\middle| \begin{matrix}\alpha_1, \ldots, \alpha_{p+q+k}\\ \beta_1, \ldots, \beta_{p+q+k}\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma}+\beta_j+\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{2\gamma+1}{4\gamma}-\beta_{q+k+j}-\frac{s}{2\gamma}\right)}$$

$$\frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma+1}{4\gamma}-\alpha_j-\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{2\gamma+1}{4\gamma}+\alpha_{q+k+j}+\frac{s}{2\gamma}\right)}\chi(s)\, x^{-s}\, ds,$$

$$\text{where } \chi(s) = \frac{\prod_{j=1}^{r} \Gamma\left(\frac{2\gamma-1}{4\gamma}+\beta_{q+j}+\frac{s}{2\gamma}\right)\prod_{j=1}^{r} \Gamma\left(\frac{2\gamma+1}{4\gamma}-\alpha_{q+j}-\frac{s}{2\gamma}\right)}{2\gamma\prod_{j=r+1}^{k} \Gamma\left(\frac{2\gamma+1}{2\gamma}-\beta_{q+j}-\frac{s}{2\gamma}\right)\prod_{j=r+1}^{k} \Gamma\left(\frac{2\gamma-1}{4\gamma}+\alpha_{q+j}+\frac{s}{2\gamma}\right)}$$

which satisfies (2.3) if

$\alpha_{q+j}+\beta_{q+j}=0$ *i. e.* $\alpha_{q+j}=-\beta_{q+j}=\sigma_j$ (say),
$j=1, \ldots, k$.

Now setting $\alpha_j=-d_j$, $\beta_j=b_j$, $j=1, \ldots, q$ and $\alpha_{q+k+j}=-a_j$, $\beta_{q+k+j}=c_j$, $j=1, \ldots, p$ it is seen that the right hand side of (3.1) becomes the same as that of (2.2).

Hence we find that

$$(3.2)\quad x^{\gamma-\frac{1}{2}}\, G^{q+r,\, q+r}_{p+q+k,\, p+q+k}\left[x^{2\gamma}\middle| \begin{matrix}-d, \ldots, -d_q, \sigma_1, \ldots, \sigma_k, -a_1, \ldots, -a_p\\ b_1, \ldots, b_q, -\sigma_1, \ldots, -\sigma_k, c_1, \ldots, c_p\end{matrix}\right],$$
$$q+2r > p+k > r,$$

is a kernel, transforming a self-reciprocal function in (1.1) into another self-reciprocal function in (1.1) with a's and b's replaced by c's and d's.

3.1. PARTICULAR CASES : Giving suitable values to the parameters in (3.2), as particular cases, we can deduce a number of known as well as unknown kernels connecting different classes of self-reciprocal functions under various generalisations of the Hankel transform in view of section 1.

With $\gamma=1$, $p=1$, $q=2$, and choosing the other parameters suitably in (3.2) we obtain the kernels due to R. K. Saxena (8, p. 68).

When $\gamma=1$, $q=1$, $p=o$ and setting the parameters suitably in (3.2) we get R. Narain's kernel (6, p. 62).

Taking $\gamma=1$ and giving suitable values to the parameters in (3.2) we can get

$$(3.3) \qquad \sqrt{x}\; G^{q+r,\, q+r}_{p+q+k,\, p+q+k}\left[x^2 \,\middle|\, \begin{matrix} -d_1, \ldots, -d_q, \sigma_1, \ldots, \sigma_k, -a_1, \ldots, -a_p \\ b_1, \ldots, b_q, -\sigma_1, \ldots, -\sigma_k, c_1, \ldots, c_p \end{matrix}\right], \quad q+2r > p+k > r,$$

as a kernel, transforming a self-reciprocal function in the transform with the kernel (1.3) into another self-reciprocal function in the same transform with a's and b's replaced by c's and d's.

If $\gamma=1$, $p=o$, $q=n$, $2\,b_j = \mu_j$ and $2\,d_j = \lambda_j$, $i=1, \ldots, n$, we have a kernel

$$(3.4) \qquad \sqrt{x}\; G^{n+r,\, n+r}_{n+k,\, n+k}\left[x^2 \,\middle|\, \begin{matrix} -\frac{\lambda_1}{2}, \ldots, -\frac{\lambda_n}{2}, \sigma_1, \ldots, \sigma_k \\ \frac{\mu_1}{2}, \ldots, \frac{\mu_n}{2}, -\sigma_1, \ldots, -\sigma_k \end{matrix}\right], \quad n+2r > k > r,$$

which transforms a self-reciprocal function in the transform with the kernel (1.4) into another self-reciprocal function in the same transform with μ's replaced by λ's.

In addition, if we put $n=2$, then the kernel (3.4) will yield

$$(3.5) \qquad \sqrt{x}\; G^{2+r,\, 2+r}_{2+k,\, 2+k}\left[x^2 \middle| \begin{matrix} -\frac{\mu'}{2}, & -\frac{\lambda'}{2}, & \sigma_1, \ldots, \sigma_k \\ \frac{\mu}{2}, & \frac{\lambda}{2}, & -\sigma_1, \ldots, -\sigma_k \end{matrix} \right],$$

$$2+2r > k > r,$$

as a kernel transforming a self-reciprocal function under the transform with the kernel (1.5) into another self-reciprocal function in the same transform with μ and λ replaced by μ' and λ' respectively.

Substituting $\gamma=k$, $q=k$, $p=o$ and $b_j=d_j=\frac{k-j}{2k}$, $j=1, \ldots, k$ me can obtain a kernel

$$(3.6) \qquad x^{k-\frac{1}{2}}\; G^{k+r,\, k+r}_{k+n,\, k+n}\left[x^{2k} \middle| \begin{matrix} o, -\frac{1}{2k}, \ldots, -\frac{k-1}{2k}, \sigma_1, \ldots, \sigma_n \\ o, \frac{1}{2k}, \ldots, \frac{k-1}{2k}, -\sigma_1, \ldots, -\sigma_n \end{matrix} \right],$$

$$k+2r > n > r,$$

transforming a self-reciprocal function in the transform with the kernel (1.6) into itself.

When $k=1$, (3.6) gives a kernel, transforming a self-reciprocal function in the Hankel transform of order zero into itself.

Putting $\gamma=k$, $p=o$, $q=k$ and $b_j=d_j=\frac{2j-1}{4k}$, $j=1, \ldots, k$, we can get

$$(3.7) \qquad x^{k-\frac{1}{2}}\; G^{k+r,\, k+r}_{k+n,\, k+n}\left[x^{2k} \middle| \begin{matrix} -\frac{1}{4k}, \ldots, -\frac{2k-1}{4k}, \sigma_1, \ldots, \sigma_n \\ \frac{1}{4k}, \ldots, \frac{2k-1}{4k}, -\sigma_1, \ldots, -\sigma_n \end{matrix} \right],$$

$$k+2r > n > r,$$

as a kernel, which transforms a self-reciprocal funetion in the transform with the kernel (1.7) into itself.

If $k=1$ (3.7) yields a kernel, which transforms a self-reciprocal function in the sine transform into itself.

4. Consider the function

$$G^{m+2q,\, m}_{n+2p,\, n+2q}\left[\beta^2 x^\rho \,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_{n+2p}\\ \beta_1, \ldots, \beta_{n+2q}\end{matrix}\right], \qquad m < n < 2m+q.$$

By definition (1, p. 207) this equals

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{2q}\Gamma(\beta_j-t)\prod_{j=1}^{m}\Gamma(\beta_{2q+j}-t)\prod_{j=1}^{m}\Gamma(1-\alpha_j+t)\,\beta^{2t}\,x^{\rho t}}{\prod_{j=1}^{2p}\Gamma(\alpha_{n+j}-t)\prod_{j=1}^{n-m}\Gamma(\alpha_{m+j}-t)\prod_{j=1}^{n-m}\Gamma(1-\beta_{2q+m+j}+t)}\,dt.$$

With $\rho=2\gamma$ and replacing t by $-\frac{s}{2\gamma}-\frac{2\gamma-1}{4\gamma}$ we get

$$(4.1)\qquad x^{\gamma-\frac{1}{2}}\, G^{m+2q,\, m}_{n+2p,\, n+2q}\left[\beta^2 x^{2\gamma} \,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_{n+2p}\\ \beta_1, \ldots, \beta_{n+2q}\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \beta^{-s/\gamma}\,\frac{\prod_{j=1}^{2q}\Gamma\left(\frac{2\gamma-1}{4\gamma}+\beta_j+\frac{s}{2\gamma}\right)}{\prod_{j=1}^{2p}\Gamma\left(\frac{2\gamma-1}{4\gamma}+\alpha_{n+j}+\frac{s}{2\gamma}\right)}\,\chi(s)x^{-s}\,ds,$$

where $\chi(s)=$

$$\frac{\prod_{j=1}^{m}\Gamma\left(\frac{2\gamma-1}{4\gamma}+\beta_{2q+j}+\frac{s}{4\gamma}\right)\prod_{j=1}^{m}\Gamma\left(\frac{2\gamma+1}{4\gamma}-\alpha_j-\frac{s}{2\gamma}\right)}{2\gamma\,\beta^{(2\gamma-1)/2\gamma}\prod_{j=1}^{n-m}\Gamma\left(\frac{2\gamma+1}{4\gamma}-\beta_{2q+m+j}-\frac{s}{2\gamma}\right)\prod_{j=1}^{n-m}\Gamma\left(\frac{2\gamma-1}{4\gamma}+\alpha_{m+j}+\frac{s}{2\gamma}\right)},$$

which satisfies the relation (2.2) if

$\alpha_j+\beta_{2q+j}=0,\ j=1, \ldots, n.$

Now putting $\beta_j = b_j$, $\beta_{q+j} = d_j$, $j=1, \ldots, q$ and $\alpha_{n+j} = -a_j$, $\alpha_{n+p+j} = -c_j$, $j=1, \ldots, p$ we see that the right hand side of (4.1) is the same as that of (2.7).

Hence we get

$$(4.2) \qquad x^{\gamma-\frac{1}{2}} G^{m+2q,\, m}_{n+2p,\, n+2q}\left[\beta^2 x^{2\gamma} \middle| \begin{matrix} \alpha_1, \ldots, \alpha_n, -a_1, \ldots, -a_p, -c_1, \ldots, -c_p \\ b_1, \ldots, b_q, d_1, \ldots, d_q, -\alpha_1, \ldots, -\alpha_n \end{matrix}\right], \quad m < n < 2m+q-p,$$

as a kernel, transforming a self-reciprocal function in (1.1) into another self-reciprocal function in (1.1) with a's and b's replaced by c's and d's and vice versa.

4.1. PARTICULAR CASES : Having suitable values of the parameters in (4.2), as particular cases, we can again deduce a number of known as well as unknown kernels, connecting different classes of self-reciprocal functions under various generalisations of the Hankel transform in view of section 1.

If $\gamma=1$, $\beta=\frac{1}{2}$, $p=1$, $q=2$ and setting the parameters suitably in (4.2) we arrive at the case due to R. K. Saxena (8, p. 66).

When $\gamma=1$, $\beta=\frac{1}{2}$, $p=0$, $q=1$ and adjusting the parameters suitably in (4.2) we have the case, given by R. Narain (6, p. 59).

With $\gamma=1$, $\beta=\frac{1}{2}$ and having suitable parameters in (4.2) we obtain

$$(4.3) \qquad \sqrt{x}\, G^{m+2q,\, m}_{n+2p,\, n+2q}\left[\frac{x^2}{4} \middle| \begin{matrix} \alpha_1, \ldots, \alpha_n, -a_1, \ldots, -a_p, -c_1, \ldots, -c_p \\ b_1, \ldots, b_q, d_1, \ldots, d_q, -\alpha_1, \ldots, -\alpha_n \end{matrix}\right], \quad m < n < 2m+q-p,$$

as a kernel, transforming a self-reciprocal function in the transform with the kernel (1.3) into another self-reciprocal function in the same transform with a's and b's replaced by c's and d's and vice versa.

When $\gamma=1$, $\beta=2^{-n}$, $p=0$, $q=n$ and $2b_j=\mu_j$, $2d_j=\gamma_j$, $j=1, \ldots, n$, from (4.2) we can deduce the kernel

$$(4.4) \qquad \sqrt{x}\, G^{s+2n,\, s}_{t,\, t+2n}\left[\frac{x^2}{4^n} \middle| \begin{matrix} \alpha_1, \ldots, \alpha_t \\ \frac{\mu_1}{2}, \ldots, \frac{\mu_n}{2}, \frac{\gamma_1}{2}, \ldots, \frac{\gamma_n}{2}, -\alpha_1, \ldots, -\alpha_t \end{matrix}\right], \quad s < t < 2s+n,$$

which transforms a self-reciprocal function in the transform with the kernel (1.4) into the same transform with μ's replaced by γ's and vice versa.

In addition, if we substitute $n=2$ the kernel (4.4) will yield

$$(4.5) \qquad \sqrt{x}\, G^{s+4,\, s}_{t,\, t+4}\left[\frac{x^2}{16} \middle| \begin{matrix} \alpha_1, \ldots, \alpha_t \\ \frac{\mu}{2}, \frac{\gamma}{2}, \frac{\mu'}{2}, \frac{\gamma'}{2}, -\alpha_1, \ldots, -\alpha_t \end{matrix}\right], \quad s < t < 2s+2,$$

as a kernel, transforming a self-reciprocal function in the transform with the kernel (1.5) into another self-reciprocal function in the same transform with μ and γ replaced by μ' and γ' respectively and vice versa.

Putting $\gamma=k$, $\beta=\left(\frac{1}{2k}\right)^k$, $p=o$, $q=k$ and $b_j=d_j=\frac{k-j}{2k}$, $j=1, \ldots, k$ in (4.2)

we obtain

$$(4.6) \qquad x^{k-\frac{1}{2}} G^{m+2k,\, m}_{n,\, n+2k}$$

$$\left[\left(\frac{x}{2k}\right)^{2k} \middle| \begin{matrix} \alpha_1, \ldots, \alpha_n \\ o, \frac{1}{2k}, \ldots, \frac{k-1}{2k}, o, \frac{1}{2k}, \ldots, \frac{k-1}{2k}, -\alpha_1, \ldots, -\alpha_n \end{matrix} \right],$$

$$m < n < 2m+k,$$

which plays the role of a kernel, transforming a self-reciprocal function in the transform with the kernel (1.6) into itself.

When $k=1$ (4.6) gives a kernel, transforming a self-reciprocal function in the Hankel transform of order zero into itself.

If we substitute $\gamma=k$, $\beta=\left(\frac{1}{2k}\right)^k$, $p=o$, $q=k$ and $b_j = d_j = \frac{2j-1}{4k}$, $j=1, \ldots,$ k in (4.2) we get a kernel

$$(4.7) \qquad x^{k-\frac{1}{2}} G^{m+2k,\, m}_{n,\, n+2k}$$

$$\left[\left(\frac{x}{2k}\right)^{2k} \middle| \begin{matrix} \alpha_1, \ldots, \alpha_n \\ \frac{1}{4k}, \ldots, \frac{2k-1}{4k}, \frac{1}{4k}, \ldots, \frac{2k-1}{4k}, -\alpha_1, \ldots, -\alpha_n \end{matrix} \right],$$

$$m < n < 2m+k,$$

which transforms a self-reciprocal function in the transform with the kernel (1.7) into itself.

With $k=1$ (4.7) yields a kernel which transforms a self-reciprocal function in sine transform into itself.

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ON GENERALIZED LAPLACE TRANSFORMS—I

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ABSTRACT

In this paper we shall establish a new theorem concerning the generalized Laplace transform introduced by Mainra (1961, p. 24) ; viz -

$$W\{f(x)\,;\,\eta+\tfrac{1}{2}\,;\,k+\tfrac{1}{2},\,r\,;\,s\}$$

$$= s\int_0^\infty (sx)^{-\eta-\frac{1}{2}}\,e^{-\frac{1}{2}sx}\,W_{k+\frac{1}{2},r}\,(sx)\,f(x)\,dx.$$

An infinite integral involving the product of H-functions has been evaluated as an application of the theorem

1. *Introduction,* Mainra (1961, p. 24) has introduced the generalization of the well-known Laplace transform

$$L\{f(x)\,;\,s\} = s\int_0^\infty e^{-sx}\,f(x)\,dx \tag{1.1}$$

by means of the integral equation

$$W\{f(x)\,;\,\eta+\tfrac{1}{2}\,;\,k+\tfrac{1}{2},\,r\,;\,s\}$$

$$= s\int_0^\infty (sx)^{-\eta-\frac{1}{2}}\,e^{-\frac{1}{2}sx}\,W_{k+\frac{1}{2},r}\,(sx)\,f(x)\,dx. \tag{1.2}$$

(1.2) reduces to (1.1), when $\eta=k=-r$ on account of the well-known identity

$$W_{\frac{1}{2}-r,r}\,(x)=x^{\frac{1}{2}-r}\quad e^{-\frac{1}{2}x} \tag{1.3}$$

If we take $\eta = k$ in (1.2), we get an integral transform given by Meijer (1941, p. 730), viz :

$$M\{f(x)\ ;\ k+\tfrac{1}{2}\ r\ ;\ s\} = s\int_0^\infty (sx)^{-k-\frac{1}{2}}\ e^{-\frac{1}{2}sx}\, W_{k+\frac{1}{2},r}\,(sx)\, f(x)\, dx \tag{1.4}$$

On the other hand if we take $\eta = -r$ and $k-\frac{1}{2}$ for k in (1.2), we get the following integral transform originated by Varma (1951, p. 209) :

$$V\{f(x)\ ;\ k, r\ ;\ s\} = s\int_0^\infty (sx)^{r-\frac{1}{2}}\ e^{-\frac{1}{2}sx}\, W_{k,r}\,(sx)\, f(x)\, dx \tag{1.5}$$

In this paper we shall be establishing a general theorem for the transform defined by (1.2). Some of the results obtained earlier by Saxena (1960, p. 404 ; 1961, p. 292) and Sharma (1964, p. 362) follow as special cases of our findings.

2. *The H-function* ; We shall represent and define the H-function introduced by Fox (1961, p. 408) in the following form as given by Gupta (1965, p. 98) :

$$H^{m,\,n}_{p,\,q}\left[\, x \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j\,\xi)\ \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j\,\xi)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j\,\xi)\ \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j\,\xi)}\ x^{\xi}\, d\xi \tag{2.1}$$

Where x is not equal to zero and an empty product is interpreted as unity ; p, q, n, m are integers satisfying $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$;

α_j ($j = 1, \ldots\ldots, p$), β_j ($j = 1, \ldots\ldots, q$) are positive numbers and

a_j ($j = 1, \ldots\ldots, p$), b_j ($j = 1, \ldots\ldots, q$) are complex numbers, such that no

pole of $\Gamma(b_h - \beta_h \xi)$ $(h=1, \ldots, m)$, coincides with any pole of $\Gamma(1-a_j+\alpha_i \xi)$ $(i=1, \ldots, n,)$ i.e.

$$\alpha_i (b_h + v) \neq \beta_h (a_i - \eta - 1) \tag{2.2}$$

$(v, \eta = 0, 1, \ldots; h=1, \ldots, m \;;\; i=1, \ldots, n)$.

Further the contour L runs from $\sigma - i\infty$ to $\sigma + i\infty$ such that the points

$$\xi = (b_h + v)/\beta_h \;(h=1, \ldots, m \;;\; v=0, 1, \ldots ;)$$

which are poles of $\Gamma(b_h - \beta_h \xi)$, lie to the right and the points

$$\xi = (a_i - \eta - 1)/\alpha_i \;(i=1; \ldots, n \;;\; \eta=0, 1, \ldots ;)$$

which are poles of $\Gamma(1-a_i+\alpha_i \xi)$, lie to the left of L. Such a contour is possible on account of (2.2).

3. The following results obtained by Gupta (1966, pp. 4, 10, 11 ; 1965, p. 99) will be used in our subsequent discussions. In what follows N and S will always stand for positive integers :

$$H^{m,n}_{p,q}\left[x \middle| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \beta_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right]$$

$$= c\, H^{m,n}_{p,q}\left[x^c \middle| \begin{matrix} (a_1, c\,\alpha_1), \ldots, (a_p, c\,\alpha_p) \\ (b_1, c\,\beta_1), \ldots, (b_q, c\,\beta_q) \end{matrix} \right] \tag{3.1}$$

where $c > 0$.

$$x^\sigma \; H^{m,n}_{p,q}\left[x \middle| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right]$$

$$= H^{m,n}_{p,q}\left[x \middle| \begin{matrix} (a_1+\sigma\,\alpha_1, \alpha_1), \ldots, (a_p+\sigma\,\alpha_p, \alpha_p) \\ (b_1+\sigma\,\beta_1, \beta_1), \ldots, (b_q+\sigma\,\beta_q, \beta_q) \end{matrix} \right], \tag{3.2}$$

$$H^{m,n}_{p,q}\left[x^{-1} \middle| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right]$$

$$= H^{n,m}_{q,p}\left[x \middle| \begin{matrix} (1-b_1, \beta_1), \ldots, (1-b_q, \beta_q) \\ (1-a_1, \alpha_1), \ldots, (1-a_p, \alpha_p) \end{matrix} \right] \tag{3.3}$$

$$x^{l}\, e^{-\frac{1}{2}x} W_{k,r}(x) = H^{2,0}_{1,2}\left[x \,\middle|\, \begin{matrix} (l-k+1,\ 1) \\ (l-r+\frac{1}{2}, 1), (l-r+\frac{1}{2}, 1) \end{matrix} \right] \tag{3.4}$$

$$H^{m,\, n+2}_{p+2,\, q+1}\left[x \,\middle|\, \begin{matrix} (l_1, S), (l_2, S), (a_1, N), \ldots, (a_p, N) \\ (b_1, N), \ldots, (b_q, N), (f, S) \end{matrix} \right]$$

$$= S^{\frac{1}{2}+f-l_1-l_2}\; N^{\sum_1^q (b_j) - \sum_1^p (a_j) + \frac{1}{2}p - \frac{1}{2}q}\; (2\pi)^{\frac{1}{2}(1-S) + (1-N)(m+n-\frac{1}{2}p-\frac{1}{2}q)}$$

$$\times\, G^{Nm,\, 2S+Nn}_{2S+Np,\, Nq+S}\left[x\, \frac{S\, N^{(p-q)}}{S\, N} \,\middle|\, \begin{matrix} \triangle(S, l_1), \triangle(S, l_2), \triangle(N, a_1), \ldots, \triangle(N, a_p) \\ \triangle(N, b_1), \ldots, \triangle(N, b_q), b_q), \triangle(S, f) \end{matrix} \right]$$

where the symbol $\triangle(S, a)$ represents the set of parameters $\frac{a}{S}, \frac{a+1}{S}, \ldots\ldots,$

$$\frac{a+S-1}{S}. \tag{3.5}$$

$$W\left\{ x^{l} H^{m,n}_{p,q}\left[z\, x^{\sigma} \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right]; \eta+\tfrac{1}{2}\, ; k+\tfrac{1}{2}, r\, ; s \right\}$$

$$= s^{-l} H^{m,\, n+2}_{p+2,\, q+1}\left[z\, s^{-\sigma} \,\middle|\, \begin{matrix} (\eta-l+r,\ \sigma), (\eta-l-r,\ \sigma), (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q), (\eta+k-l,\ \sigma) \end{matrix} \right] \tag{3.6}$$

provided that $\sigma > 0$, $R(s) > 0$, $R(l-\eta \pm r+1+\sigma \min b_h/\beta_h) > 0$, and the set of conditions given below is satisfied :

(i) $\lambda > 0$ and $|\arg(z)| < \frac{1}{2}\lambda\pi$

(ii) $\lambda = 0$, z is real and positive and $R(\mu) < 0$,

where λ and μ stand for the quantities

$$\sum_{1}^{n}(\alpha_j) - \sum_{n+1}^{p}(\alpha_j) + \sum_{1}^{m}(\beta_j) - \sum_{m+1}^{p}(\beta_j) \qquad \text{and}$$

$$\sum_{1}^{q} (b_j) - \sum_{1}^{p} (a_j) + \tfrac{1}{2}(p-q)$$

respectively.

4. *Theorem* 1.

If

$$f(s) = W\{ g(x)\,\phi(x) \,;\, \delta+\tfrac{1}{2} \,;\, \lambda+\tfrac{1}{2}, \mu \,;\, s \} \tag{4.1}$$

and

$$s^{l}\,\phi(s^{\sigma}) = W\{ h(x) \,;\, \eta+\tfrac{1}{2} \,;\, k+\tfrac{1}{2}, r \,;\, s \} \tag{4.2}$$

then

$$f(s) = \sigma\, s^{\frac{1}{2}-\delta}$$

$$\int_0^\infty \frac{h(t)}{t}\, W\{ x^{\sigma(\frac{1}{2}-\delta)-l}\, g(x^{\sigma})\, e^{-\frac{1}{2}sx^{\sigma}}\, W_{\lambda+\frac{1}{2},\,\mu}(sx^{\sigma}) \,;\, \eta+\tfrac{1}{2} \,;\, k+\tfrac{1}{2}, r \,;\, t \}\, dt, \tag{4.3}$$

provided that the integral is convergent, $R(s) > 0$, $\sigma > 0$ and the Mainra transform of $|\, g(x)\,\phi(x) \,|$ and $|\, h(x) \,|$ exist.

Proof :—We have from (4.2)

$$\phi(s^{\sigma}) = s^{1-l} \int_0^\infty (st)^{-\eta-\frac{1}{2}}\, e^{-\frac{1}{2}st}\, W_{k+\frac{1}{2},\, r}(st)\, h(t)\, dt$$

Therefore,

$$f(s) = s \int_0^\infty (sx)^{-\delta-\frac{1}{2}}\, e^{-\frac{1}{2}sx}\, W_{\lambda+\frac{1}{2},\,\mu}(sx)\, g(x)$$

$$\times \left[x^{\frac{1-l}{\sigma}} \int_0^\infty (x^{1/\sigma} t)^{-\eta-\frac{1}{2}}\, e^{-\frac{1}{2}x^{1/\sigma}\cdot t}\, W_{k+\frac{1}{2},r}(x^{1/\sigma}t)\, h(t)\,dt \right] dx. \tag{4.4}$$

On inverting the order of integration in (4.4), which is easily seen to be permissible by virtue of De La Vallèè Poussins theorem (1956, p. 504) under the conditioned stated with the theorem, we get

$$f(s) = s\int_0^{\infty} h(t)\, t^{-\eta-\frac{1}{2}} \left[\int_0^{\infty} (sx)^{-\delta-\frac{1}{2}}\; e^{-\frac{1}{2}sx}\; x^{(\frac{1}{2}-l-\eta)/\sigma}\, W_{\lambda+\frac{1}{2},\mu}(sx) \right.$$

$$\left. \times \left\{ e^{-\frac{1}{2}x^{1/\sigma}\cdot t} W_{k+\frac{1}{2},r}\,(x^{1/\sigma}.\,t) g(x) \right\} dx \right] dt. \tag{4.5}$$

Putting $x^{1/\sigma} = u$ in (4.5), and interpreting the result thus obtained by the help of (1.2), we get the required result.

Since the theorem proved above involves a generel function $g(x)$, many interesting theorems can be obtained from it by specializing this function.

5. *Theorem* 1(*a*). If we take $g(x) = x^{c}$ in theorem 1, it reduces to the following form : If

$$f(s) = W\{x^{c}\;\phi(x)\,;\,\delta+\tfrac{1}{2}\,;\,\lambda+\tfrac{1}{2},\,\mu\,;\,s\},$$

and

$$s^{l}\,\phi(s^{\sigma}) = W\{h(x)\,;\,\eta+\tfrac{1}{2}\,;\,k+\tfrac{1}{2},\,r\,;\,s\},$$

then

$$f(s) = \sigma . s^{l/\sigma - c} \int_0^{\infty} \frac{h(t)}{t} H^{2,2}_{3,3}$$

$$\left[st^{-\sigma} \middle| \begin{matrix} (\eta+r,\,\sigma),\,(\eta-r,\,\sigma)\,(1+c-\delta-\lambda-\frac{l}{\sigma},\,1) \\ (1+c-\delta+\mu-\frac{l}{\sigma},\,1),\,(1+c-\delta-\mu-\frac{l}{\sigma},\,1),\,(\eta+k,\,\sigma) \end{matrix} \right] dt \tag{5.1}$$

provided that the integral is convergent, $R(s) > 0$, $\sigma > 0$ and the Mainra transform of $| x^c \phi(x) |$ and $| h(x) |$ exist.

The above theorem easily follows by virtue of the results (3.4) and (3.6) after a little simplification.

Example. On taking

$$h(x) = H^{m,n}_{p,q}\left[zx^{\sigma}\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right.\right] \tag{5.2}$$

in theorem 1(a), we have by virtue of (3.6)

$$s^l \phi(s^{\sigma}) = H^{m,n+2}_{p+2,q+1}\left[\frac{z}{s^{\sigma}}\left|\begin{matrix}(\eta+r, \sigma), (\eta-r, \sigma), (a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(b_1, \beta_1), \ldots, (b_q, \beta_q), (\eta+k, \sigma)\end{matrix}\right.\right], \tag{5.3}$$

where $\sigma > 0$, $R(s) > 0$, $R\left(-\eta \pm r + 1 + \sigma \min \frac{b_h}{\beta_h}\right) > 0$,

$$\lambda = \left\{\sum_1^n (\alpha_j) - \sum_{n+1}^p (\alpha_j) + \sum_1^m (\beta_j) - \sum_{m+1}^q (\beta_j)\right\} > 0 \text{ and}$$

$| \arg(z) | < \frac{1}{2}\lambda\pi.$

Therefore, by virtue of (3.3), we have

$x^c \phi(x)$

$$= x^{c-l/\sigma} H^{n+2,m}_{q+1,p+2}\left[\frac{x}{z}\left|\begin{matrix}(1-b_1, \beta), \ldots, (1-b_q, \beta_q), (1-\eta-k, \sigma)\\(1-\eta+r, \sigma), (1-\eta-r, \sigma), (1-a_1, \alpha_1), \ldots, (1-a_p, \alpha_q)\end{matrix}\right.\right], \tag{5.4}$$

on applying (3.6) again in the equation (5.4), we get, under the conditions easily obtainable from (3.6)

f (s)

$$= s^{l/\sigma - c}\, H^{n+2,\, m+2}_{q+3,\, p+3}$$

$$\left[\frac{1}{zs}\left|\begin{matrix}(\delta+\frac{l}{\sigma}-c+\mu,\ 1),\ (\delta+\frac{l}{\sigma}-c-\mu,\ 1),\ (1-b_1,\ \beta_1),\ \ldots,\ (1-b_q,\ \beta_q),\ (1-\eta-k,\ \sigma)\\ (1-\eta+r,\ \sigma),\ (1-\eta-r,\ \sigma),\ (1-a_1,\ \alpha_1),\ \ldots,\ (1-a_p,\ \alpha_p),\ (\delta+\lambda+\frac{l}{\sigma}-c,\ 1)\end{matrix}\right.\right] \tag{5.5}$$

substituting the above values of $h\,(x)$ and $f\,(s)$ in (5.1), we get the following integral by virtue of the equations (3.2) and (3.3) after a little simplification

$$\int_0^\infty H^{m,n}_{p,q}\left[zx\left|\begin{matrix}(a_1,\ \alpha_1),\ \ldots,\ (a_p,\ \alpha_p)\\ (b_1,\ \beta_1),\ \ldots,\ (b_q,\ \beta_q)\end{matrix}\right.\right]$$

$$\times\, H^{2,2}_{3,3}\left[\frac{x}{z}\left|\begin{matrix}\left(\frac{l}{\sigma}+\delta-c+\mu-1,\ 1\right),\ \left(\frac{l}{\sigma}+\delta-c+\mu-1,\ 1\right)(1-\eta-k-\sigma,\ \sigma)\\ (1-\eta+r-\sigma,\ \sigma),\ (1-\mu-r-\sigma,\ \sigma),\ (\delta+\lambda+\frac{l}{\sigma}-c-1,\ 1)\end{matrix}\right.\right] dx$$

$$= s\, H^{n+2,\, m+2}_{q+3,\, p+3}$$

$$\left[\frac{1}{zs}\left|\begin{matrix}(\delta+\frac{l}{\sigma}-c+\mu,\ 1),\ (\delta+\frac{l}{\sigma}-c-\mu,1),\ (1-b_1,\ \beta_1),\ \ldots,\ (1-b_q,\ \beta_q),\ (1-\eta-k,\ \sigma)\\ (1-\eta+r,\ \sigma),\ (1-\eta-r,\ \sigma),\ (1-a_1,\ \alpha_1),\ \ldots,\ (1-a_p,\ \alpha_p),\ (\delta+\lambda+\frac{l}{\sigma}-c\ 1)\end{matrix}\right.\right] \tag{5.6}$$

provided that the following conditions are satisfied

$$\sigma > o,\ R\,(s) > o,\ R\left[\min\frac{b_h}{\beta_h}+\frac{1-\eta\pm r-\sigma}{\sigma}+1\right] > o\ (h=1,\ \ldots,\ m)\ ;$$

and $R\left[\left(\frac{1-a_j}{\alpha_j}\right)+3+c-\delta\pm\mu-\frac{l}{\sigma}\right] < o\ (j=1, .., n)$.

Further on putting $\alpha_j = \beta_h\ (j=1, ..., p, ..., h=1, ..., q) = \sigma = 1$ in (5.6), we get a particular case of the known integral (2, p. 422).

Corollary 1. On putting $\sigma = \frac{N}{S}$ in theorem 1(*a*), it reduces to the following form involving Meijer's G-function by virtue of the equations (3.1) and (3.5).

If

$$f(s) = W\{x^c\ \phi(s)\ ;\ \delta+\tfrac{1}{2}\ ;\ \lambda+\tfrac{1}{2},\ \mu\ ;\ s\},$$

and

$$s^l\ \phi(s^{N/S}) = W\{h(x)\ ;\ \eta+\tfrac{1}{2}\ ;\ k+\tfrac{1}{2},\ r\ ;\ s\},$$

then

$$f(s) = s^{lS/N-c}\ N^{3/2+k-n}\ S^{\frac{1}{2}+c-\delta+\lambda-lS/N}\ (2\pi)^{\frac{1}{2}(2-N-S)}$$

$$\times\int_0^\infty \frac{h(t)}{t}\, G^{2s,2N}_{2N+S,2S+N}$$

$$\left[\frac{s^S\ N^N}{S^s\ t^N}\left|\begin{matrix}\Delta(N,\eta+r),\ \Delta(N,\eta-r),\ \Delta(S,\ 1+c-\delta-\lambda-\frac{lS}{N})\\ \Delta(S,\ 1+c-\delta+\mu-\frac{lS}{N}),\ \Delta(S,\ 1+c-\delta-\mu-\frac{lS}{N}),\ \Delta(N,\eta+k)\end{matrix}\right.\right]dt, \tag{5.7}$$

provided that the integral is convergent, $R(s) > 0$ and the Mainra transform of $|x^c\ \phi(x)|$ and $|h(x)|$ exist.

Corollary 2. On taking $\delta=\lambda$ and $\eta=k$ in corollary 1, we arrive at the following result :

If

$$f(s) = \mathrm{M}\{x^{c}\ \phi(x)\,;\ \lambda+\tfrac{1}{2},\ \mu\,;\ s\},$$

and

$$s^{l}\ \phi\ (s^{\mathrm{N}/s}) = \mathrm{M}\{h(x)\,;\ k+\tfrac{1}{2},\ r\,;\ s\},$$

then

$$f(s) = s^{l\mathrm{S}/\mathrm{N}-c}\ \mathrm{N}^{3/2}\ \mathrm{S}^{\frac{1}{2}+c-l\mathrm{S}/\mathrm{N}}\ (2\pi)^{\frac{1}{2}(2-\mathrm{N}-\mathrm{S})}$$

$$\times\int_{0}^{\infty}\frac{h(t)}{t}\,G^{2\mathrm{S},\,2\mathrm{N}}_{2\mathrm{N}+\mathrm{S},\,2\mathrm{S}+\mathrm{N}}\left[\frac{s^{\mathrm{S}}\ \mathrm{N}^{\mathrm{N}}}{\mathrm{S}^{s}\ t^{\mathrm{N}}}\left|\begin{matrix}\Delta(\mathrm{N},k+r),\ \Delta(\mathrm{N},k-r),\ \Delta\left(\mathrm{S},\ 1+c-2\lambda-\frac{ls}{\mathrm{N}}\right)\\ \Delta\left(\mathrm{S},\ 1+c-\lambda-\frac{l\mathrm{S}}{\mathrm{N}}+\mu\right),\ \Delta\left(\mathrm{S},\ 1+c-\lambda-\frac{ls}{\mathrm{N}}-\mu\right),\ \Delta(\mathrm{N},2k)\end{matrix}\right.\right]dt,$$

(5.8)

provided that the integral is convergent, $R(s) > 0$, and the Meijer transform of $|\,x^{c}\ \phi(x)\,|$ and $|\,h(x)\,|$ exist.

Corollary 3. On taking $l = -\frac{\sigma\mathrm{N}}{\mathrm{S}}$, $\eta = -m$, $r = m$, $c = -\rho-\sigma$, $\delta = -\rho$ and further putting $k=\frac{1}{2}$ for k and $\lambda=\frac{1}{2}$ for λ in corollary 1, we get the result obtained earlier by Saxena (1960, p. 404).

6. *Theorem* 1(*b*). On taking $h=k=-r$ in theorem 1, it reduces to the following interesting theorem :

If

$$f(s) = \mathrm{W}\{ g(x)\, \phi(x) ; \delta+\tfrac{1}{2} ; \lambda+\tfrac{1}{2}, \mu ; s \},$$

and

$$s^{l}\, \phi(s^{\sigma}) = \mathrm{L}\{ h(x) ; s \},$$

then

$$f(s) = \sigma\, s^{\frac{1}{2}-\delta} \int_0^{\infty} \frac{h(t)}{t}\, \mathrm{L}\, \{ x^{\sigma(\frac{1}{2}-\delta)-l}\, e^{-\frac{1}{2}sx^{\sigma}}\, g(x^{\sigma})\, \mathrm{W}_{\lambda+\frac{1}{2},\mu}(sx^{\sigma}) ; t\}\, dt, \tag{6.1}$$

provided that the integral is convergent, $\mathrm{R}(s) > 0$, $\sigma > 0$ and the Mainra transform of $| g(x)\, \phi(x) |$ and the Laplace transform of $| h(x) |$ exist.

Corollary 1. On taking $g(x) = x^{c}\, e^{-ax^{l/\sigma}}$ in theorem 1(*b*) we get the following result by virtue of (3.4), (3.6) and the well-known property of Operational Culculus, viz :

If

$$\phi(s) = \mathrm{L}\, [h(x) ; s],$$

then

$$\frac{s}{s+a}\, \phi(s+a)\, \mathrm{L}\left[e^{-ax}\, h(x) ; s \right].$$

If

$$f(s) = \mathrm{W}\, \{x^{c}\, e^{-ax^{1/\sigma}}\, \phi(x) ; \delta+\tfrac{1}{2} ; \lambda+\tfrac{1}{2}, \mu ; s \}$$

and

$$s^{l}\ \phi\ (s^{\sigma}\ ) = \mathrm{L}\ \{\ h\ (x)\ ;\ s\}$$

then

$$f\ (s) = \sigma\ s^{l/\sigma - c} \int_{0}^{\infty} \frac{h\ (t)}{t+a}\ \mathrm{H}_{2,2}^{2,1}$$

$$\left[\frac{s}{(t+a)^{\sigma}}\ \middle|\ \begin{matrix}(0,\sigma),\ (1-\delta+e-\lambda-\frac{l}{\sigma},\ 1 \\ (1-\delta+c+\mu-\frac{l}{\sigma},\ 1),\ (1-\delta+c-\mu-\frac{l}{\sigma},\ 1)\end{matrix}\right] dt \tag{6.2}$$

provided that the integral is convergent, $R\ (s) > 0$, $\sigma > 0$ and the Mainra transform of $|\ x^{c}\ e^{-ax^{1/\sigma}}\ \phi\ (x)\ |$ and the Laplace transform of $|\ h\ (x)\ |$ exist.

Corollary 2. On taking $\sigma = \frac{\mathrm{N}}{\mathrm{S}}$ in corollary 1, we get the following result by virtue of the equations (3.1) and (3.5) after a little simplification :

If

$$f\ (s) = \mathrm{W}\ \{\ x^{c}\ e^{-ax^{\mathrm{S/N}}}\ \phi\ (x)\ ;\ \delta+\tfrac{1}{2}\ ;\ \lambda+\tfrac{1}{2},\ \mu\ ;\ s\ \},$$

and

$$s^{l}\ \phi\ (s^{\mathrm{N/S}}\ ) = \mathrm{L}\ \{\ h\ (x)\ ;\ s\ \},$$

then

$$f\ (s) = \mathrm{N}^{3/2}\ s^{l\mathrm{S/N}-c}\ \mathrm{S}^{\frac{1}{2}-\delta+c+\lambda-l\mathrm{S/N}}\ (2\pi)^{\cdot}\ (2-\mathrm{N}-\mathrm{S})$$

$$\times \int_0^\infty \frac{h(t)}{(t+a)} G^{2S,N}_{N+S,2S}$$

$$\left[\frac{N^N s^S}{S^s (t+a)^N} \middle| \begin{matrix} \Delta(N,0), \Delta(S, 1-\delta+c-\lambda-\frac{lS}{N}) \\ \Delta(S, 1-\delta+c+\mu-\frac{lS}{N}), \Delta(S, 1-\delta+c-\mu-\frac{lS}{N}) \end{matrix} \right] dt, \tag{6 3}$$

provided that the integral is convergent, $R(s) > 0$, and the Mainra transform of $| x^c e^{-ax^{S/N}} \phi(x) |$ and the Laplace transform of $| h(x) |$ exist.

Corollary 3. If we take $\delta=\lambda$ in corollary 2 it reduces to the following form.

If

$$f(s) = M\{ x^c e^{-ax^{S/N}} \phi(x) ; \lambda+\tfrac{1}{2}, \mu ; s \},$$

and

$$s^l \phi(s^{N/S}) = L\{ h(x) ; s \},$$

then

$$f(s) = N^{3/2} s^{lS/N-c} S^{\frac{1}{2}+c-lS/N} (2\pi)^{\frac{1}{2}(2-N-S)}$$

$$\times \int_0^\infty \frac{h(t)}{(t+a)} G^{2S,N}_{N+S,2S}$$

$$\left[\frac{N^N s^S}{S^S (t+a)^N} \middle| \begin{matrix} \Delta(N,0), \Delta(S, 1-2\lambda+c-\frac{lS}{N}) \\ \Delta(S, 1-\lambda+c+\mu-\frac{lS}{N}), \Delta(S, 1-\lambda+c-\mu-\frac{lS}{N}) \end{matrix} \right] dt, \tag{6.4}$$

provided that the integral is convergent, $R(s) > 0$, and the Meijer transform of $| x^c e^{-ax^{S/N}} \phi(x) |$ and the Laplace transform of $| h(x) |$ exist.

Corollary 4. On taking $l=0$, $c=-l$, $\delta=-\mu$ and further putting $\lambda=\frac{1}{2}$ for λ in corollary 2, we arrive at another result given by Saxena (1961, p. 287).

Corollary 5. On putting $\delta=\lambda=-\mu$, $\sigma=1$, and $l=1$, in theorem 1(b) we get a result due to Sharma (1964, p. 362).

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A STUDY OF GAUSS HYPERGEOMETRIC TRANSFORM

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ABSTRACT

The purpose of this note is to study a few relationships existing between the new Gauss-Hypergeometric transform, Stieltjes and Varma transforms. This has been done in three theorems. These theorems include as particular cases a few results recently obtained by R. K. Saxena and S. C. Arya.

Introduction : In the year 1951, Varma (9, p. 209) introduced a generalization of the Laplace integral

$$\phi(p) = \int_0^\infty e^{-pt} h(t)\, dt, \quad \ldots \quad \ldots \quad (1.1)$$

in the form

$$g(p\,;k,m) = \int_0^\infty e^{-\frac{1}{2}p} \; W_{k,m}(pt)\,(pt)^{m-\frac{1}{2}}\, h(t)\, dt, \qquad (1.2)$$

where $W_{k,m}(z)$ is the Whittaker function.

In the same paper he obtained a generalization of the Stieltjes transform also. He showed that if $g(p\,;k,m)$ be the Varma transform of $h(t)$ given by (1.2) and $h(p)$ be the Laplace transform of a function $f(t)$, then

$$g(p\,;k,m) = \frac{\Gamma(2m+1)}{p\,\Gamma(m-k+3/2)} \int_0^\infty \left({2m+1,\ 1 \atop m-k+3/2} \ ;\ -\frac{t}{p} \right) f(t)\, dt \qquad (1.3)$$

$R(2m+1) > 0.$

which is the generalized form of a Stietjes transform. Recently Swaroop (8, p. 209) generalized this integral equation (1.3) by introducing three independent parameters intsead of two. His generalization is,

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$$\phi\left(\begin{matrix}p;\lambda,\mu\\ \nu\end{matrix}\right)=\frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\nu)\,p}\int_0^\infty F\left(\begin{matrix}\lambda,\mu\\ \nu\end{matrix};-\frac{t}{p}\right)f(t)\,dt \quad .. \quad (1.4)$$

He called it as "Gauss Hypergeometric Transform".

When $\mu=\nu$, (1.4) gives,

$$\phi_1(p;\lambda)=\frac{p^{2-\lambda}}{\Gamma(\lambda)}\left\{\phi\left(\begin{matrix}p;\lambda,\mu\\ \nu\end{matrix}\right)\right\}_{\mu=\nu}=p\int_0^\infty \frac{f(t)\,dt}{(p+t)^\lambda}\ \frac{s}{\lambda}\!=f(t) \quad (1.5)$$

which is a generalized Stieltjes transform of $f(t)$.

If we replace p by $\frac{\lambda}{s}$ in (1.4) and let $\lambda\to\infty$, we obtain

$$\phi_2(s;\mu,\nu)=\lim_{\lambda\to\infty}\frac{\lambda}{s\,\Gamma(\lambda)}\left\{\phi\left(\begin{matrix}p;\lambda,\mu\\ \nu\end{matrix}\right)\right\}_{p=\frac{\lambda}{s}}$$

$$=\frac{\Gamma(\mu)}{\Gamma(\nu)}\int_0^\infty {}_1F_1(\mu;\nu;-st)\,f(t)\,dt, \quad ... \quad (1.6)$$

and this may be looked as ${}_1F_1$ transform of $f(t)$.

When $\mu=\nu$ (1.6) reduces to,

$$\phi_3(s)=\lim_{\lambda\to\infty}\frac{\lambda}{s\,\Gamma(\lambda)}\left\{\phi\left(\begin{matrix}p;\lambda,\mu\\ \nu\end{matrix}\right)\right\}_{\substack{p=\frac{\lambda}{s}\\ \mu=\nu}}=\int_0^\infty e^{-st}f(t)\,dt. \quad ... \quad (1.7)$$

which is the Laplace transform of $f(t)$.

We shall denote the p-multiplied Laplace and Varma transform as $\phi_3(s)\doteq f(t)$

and $g(p\,;k\,m) \overset{V}{\underset{k,m}{=\!=}} f(t)$ respectively. (1.4) will be denoted as

$$\mathrm{F}\left\{ f(t)\,;\, {\lambda,\mu \atop v} \right\} = \phi(p).$$

2. Here we mention the Gauss hypergeometric transform of certain functions, easily obtainable from [2, p. 209, 221, 222] and [5, p. 401].

$$\mathrm{F}\left\{ t^{\sigma-1}\, G^{\alpha,\,\beta}_{\gamma,\,\delta}\left(zt^{n/s} \,\middle|\, {a_1,\ldots,a_\gamma \atop b_1,\ldots,b_\delta} \right);\, {\lambda,\mu \atop v} \right\} = (2\pi)^{(1-n)+(1-s)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)} \times$$

$$n^{\lambda+\mu-v-1}\; s^{\sum_{i=1}^{\delta} b_i - \sum_{i=1}^{\gamma} a_i + \frac{1}{2}\gamma - \frac{1}{2}\delta + 1}\; p^{\sigma-1} \times$$

$$G^{s\alpha+2n,\; s\beta+n}_{s\gamma+2n,\; s\delta+2n}\left(\frac{z^s p^n}{s^{s(\delta-\gamma)}} \,\middle|\, {\Delta(s\,;a_1),\ldots,\Delta(s\,;a_\beta),\Delta(n\,;1-\sigma),\Delta(n\,;v-\sigma),\Delta(s\,;a_{\beta+1}),\ldots,\Delta(s\,;a_\gamma), \atop \Delta(s\,;b_1),\ldots,\Delta(s\,;b_\alpha),\Delta(n\,;\lambda-\sigma),\Delta(n\,;\mu-\sigma),\Delta(s\,;b_{\alpha+1}),\ldots,\Delta(s\,;b_\delta)} \right) \tag{2.1}$$

where n and s positive integers $\Delta(n\,;\alpha)$ stands for $\frac{\alpha}{n},\ldots,\frac{\alpha+n-1}{n}$

$$0 \leqslant \alpha \leqslant \delta,\; 0 \leqslant \beta \leqslant \gamma,\; |\arg z| < (\alpha+\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta)\,\pi,\; \mathrm{R}(p) > 0$$

$$(\alpha+\beta) > \tfrac{1}{2}(\gamma+\delta),\; \mathrm{R}\left(\frac{n}{s}\min b_j\right) > \mathrm{R}(-\sigma) > \mathrm{R}\left(\frac{n}{s}a_f - \lambda\ (\text{or } \mu) - \frac{n}{s}\right),$$

$\{j=1,\ldots,\alpha.\,; f=1,\ldots,\beta\}$.

Since G-function is the generalization of a great many of special functions occuring in applied mathematics, (2.1) can yield results involving Bessel, Legendre, Whittaker and other related functions. However, we give below some particular cases of (2.1) obtained by giving suitable values to its parameters. In what follows n and s are positive integers, $\triangle(n; \alpha)$ stands for $\frac{\alpha}{n}, \ldots, \frac{\alpha+n-1}{n}$.

If $\mu = \nu$ we obtain,

$$t^{\sigma-1} G^{\alpha,\beta}_{\gamma,\delta}\left(zt^{n/s} \left| \begin{matrix} a_1, \ldots, a_\gamma \\ b_1, \ldots, b_\delta \end{matrix} \right.\right) \underset{\lambda}{\overset{S}{=}} \frac{p^{\sigma-\lambda+1}}{\Gamma(\lambda)} (2\pi)^{(1-n)+(1-s)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)}$$

$$s^{\sum_{i=1}^{\sigma} b_i - \sum_{i=1}^{\gamma} a_i + \frac{1}{2}\gamma - \frac{1}{2}\delta + 1} \; n^{\lambda-1} \times$$

$$G^{s\alpha+n,\, s\beta+n}_{s\gamma+n,\, s\delta+n}\left(\frac{z^s p^n}{s^{s(\delta-\gamma)}} \left| \begin{matrix} \triangle(s; a_1), \ldots, \triangle(s; a_\beta), \triangle(n; 1-\sigma), \triangle(s; a_{\beta+1}), \ldots, \triangle(s; a_\gamma) \\ \triangle(s; b_1), \ldots, \triangle(s; b_\alpha), \triangle(n; \lambda-\sigma), \triangle(s; b_{\alpha+1}), \ldots, \triangle(s; b_\delta) \end{matrix} \right.\right) \quad \ldots(2.2)$$

where, $R(p) > 0$, $0 \leqslant \alpha \leqslant \delta$, $0 \leqslant \beta \leqslant \gamma$, $\alpha+\beta > \frac{1}{2}(\gamma+\delta)$,

$$|\arg z| < (\alpha+\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta)\pi, \; R\left(\frac{n}{s}\min b_j\right) > R(-\sigma) > R\left(\frac{n}{s}a_f - \lambda - \frac{n}{s}\right).$$

$\{j=1, \ldots, \alpha; f=1, \ldots, \beta\}$.

when $s=1$ (2.2) reduces to a known result due to Saxen [6, p. 341]

and if $n=1$ also it reduces to a known result [3, p. 419].

$$F\left\{t^{\sigma-1} e^{-\frac{1}{2}zt^{n/s}} W_{k,m}\left(zt^{n/s}\right); \begin{matrix}\lambda, \mu \\ \nu\end{matrix}\right\} = p^{\sigma-1} (2\pi)^{(1-n)+\frac{1}{2}(1-s)} \, n^{\lambda+\mu-\nu-1} \times$$

$$s^{k+\frac{1}{2}}\, G^{2s+2n,\ n}_{s+2n,\ 2s+2n}\left(\frac{z^s p^n}{s^s} \middle| \begin{matrix} \Delta(n;1-\sigma),\ \Delta(n;\nu-\sigma),\ \Delta(s;1-k) \\ \Delta(n;\lambda-\sigma),\ \Delta(n:\mu-\sigma)\ \Delta(s;\tfrac{1}{2}+m),\ \Delta(s;\ -m) \end{matrix} \right) \quad \ldots \quad (2.3)$$

where, $R(p) > O$, $|\arg z| < \frac{\pi}{2}$, $R\{\frac{n}{s}(\frac{1}{2} \pm m)\} > R(-\sigma)$.

If $\mu=\nu$, then

$$t^{\sigma-1}\, e^{-\frac{1}{2} zt^{n/s}}\, W_{k,\ m}(zt^{n/s}) \underset{\lambda}{\overset{S}{=}} (2\pi)^{(1-n)+\frac{1}{2}(1-s)}\, n^{\lambda-1}\, s^{k+\frac{1}{2}} \{\Gamma(\lambda)\}^{-1} \times$$

$$p^{\sigma-\lambda+1}\ G^{2s+n,\ n}_{s+n,\ 2s+n}\left(\frac{z^s p^n}{s^s} \middle| \begin{matrix} \Delta(n;1-\sigma),\ \Delta(s;1-k) \\ \Delta(n;\lambda-\sigma),\ \Delta(s;\tfrac{1}{2}+m),\ (s;\tfrac{1}{2}-m) \end{matrix} \right), \quad \ldots (2.4)$$

where, $R(p) > O$, $|\arg z| < \frac{\pi}{2}$, $R\{\frac{n}{s}(\frac{1}{2} \pm m)\} > R(-\sigma)$.

If $n=s=1$, we obtain a known result, [3, p. 237].

$$F\left\{ t^{\sigma-1}\, {}_2F_1\left({a,\ b \atop c} ; -zt^{n/s} \right) ; {\lambda,\ \mu \atop \nu} \right\}$$

$$= \frac{\Gamma(c)\, p^{\sigma-1}}{\Gamma(a)\, \Gamma(b)} (2\pi)^{2-n-s}\ n^{\lambda+\mu-\nu-1}\ s^{a+b-c} \times$$

$$G^{s+2n,\ 2s+n}_{2s+2n,\ 2s+2n}\left(z^s p^n \middle| \begin{matrix} \Delta(n;1-\sigma),\ \Delta(n;\nu-\sigma),\ \Delta(s;1-a),\ \Delta(s;1-b) \\ \Delta(n;\lambda-\sigma),\ \Delta(n;\mu-\sigma),\ \Delta(s;o),\ \Delta(s;1-c) \end{matrix} \right), \quad \ldots \quad (2.5)$$

where, $R(p) > O$, $|\arg z| < \pi$, $R(\sigma) > O$, and

$$R\ \frac{n}{s}(a \text{ or } b) + R\ (\lambda \text{ or } \mu) > R\ (\sigma).$$

If $\mu=\nu$, then

$$t^{\sigma-1}\ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\ ;\ zt^{n/s}\right) \overset{S}{\underset{\lambda}{=}} (2\pi)^{2-n-s}\ n^{\lambda-1}\ s^{a+b-c}\ p^{\sigma-\lambda+1}\ \{\Gamma(\lambda)\}^{-1} \times$$

$$G^{s+n,\ 2s+n}_{2s+n,\ 2s+n}\left(z^s p^n \middle| \begin{matrix} \Delta(n\ ;\ 1-\sigma),\ \Delta(s\ ;\ 1-a)\ \Delta(s\ ;\ 1-b) \\ \Delta(n\ ;\ \lambda-\sigma),\ \Delta(s\ ;\ O),\ \Delta(s\ ;\ 1-c) \end{matrix}\right), \quad \text{— (2.6)}$$

where, $R(p) > O$, $|\arg z| < \pi$, $R(\sigma) > O$

$$\left[R\ \frac{n}{s}\ (a \text{ or } b) + R\ (\lambda) \right] > R\ (\sigma).$$

3. In this section we establish three general theorems.

Theorem 1.

If

$$f(t) \overset{V}{\underset{k,\ m}{=}} p^{n\rho+n/s}\ \phi\ (p^{n/s}\ ;\ k,\ m)$$

and

$$F\left\{\phi(t)\ ;\ \begin{matrix}\lambda,\ \mu \\ \nu\end{matrix}\right\} = \psi(p)$$

then

$$\psi(p) = p^{-1}\ (2\pi)^{\frac{1}{2}(2-n-s)}\ s^{\lambda+\mu-\nu-1}\ n^{k+m-n\rho+1} \times$$

$$\int_0^\infty t^{n\rho-1}\ G^{s,\ 2s+2n}_{2s+2n,\ 2s+n}\left(\frac{n^n}{p^s\ t^n}\middle| \begin{matrix} \Delta(n\ ;\ n\rho),\ \Delta(n\ ;\ n\rho-2m),\ \Delta(s\ ;\ 1-\lambda),\ \Delta(s\ ;\ 1-\mu) \\ \Delta(2n\ ;\ 2n\rho+2k-2m-1),\ \Delta(s\ ;\ O),\ \Delta(s\ ;\ 1-\nu) \end{matrix}\right) \times f(t)\ dt. \quad (3.1)$$

where n and s are positive integers, $\Delta(n;\alpha)$ stands for $\frac{\alpha}{n}, \frac{\alpha+1}{n}, \ldots, \frac{\alpha+n-1}{n}$. $R(p) > 0$ and provided the integral is convergent, Varma transform of $|f(t)|$ and Gauss-hypergecometric transform of $|\phi(t)|$ exist.

Proof.

We have,

$$f(t) \underset{k,\, m}{\overset{V}{=}} p^{np+n/s} \, \phi\left(p^{n/s}\,;\, k,\, m\right) \qquad \ldots \qquad (3.2)$$

and [5, p. 404].

$$\frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\nu)} \, t^{-np} \, {}_2F_1\left(\begin{matrix}\lambda,\ \mu\\ \nu\end{matrix}\,;\, -zt^{n/s}\right)$$

$$\underset{k,\, m}{\overset{V}{=}} p^{np} \, (2\pi)^{\frac{1}{2}(2-n-s)} \, s^{\lambda+\mu-\nu} \, n^{k+m-np} \times$$

$$G^{s,\, 2s+2n}_{2s+2n,\, 2s+n}\left(\frac{z^s n^n}{p^n}\left|\begin{matrix}\Delta(n\,;\, np),\ \Delta(n\,;\, np-2m),\ \Delta(s\,;\, 1-\lambda),\ \Delta(s\,;\, 1-\mu)\\ \Delta(2n\,;\, 2np+2k-2m-1),\ \Delta(s\,;\, O)\ \Delta(s\,;\, 1-\nu)\end{matrix}\right.\right), \qquad \ldots \qquad (3.3)$$

where, $R(p) > 0$, $|\arg z| < \pi$, $R(1-np+m\pm m) > 0$.

Applying the generalized Parseval-Goldstein theorem [4, p. 66] to these relations we obtain,

$$\frac{\Gamma(\lambda)\ \Gamma(\mu)}{(\Gamma\nu)} \int_0^\infty t^{-np-1} \, {}_2F_1\left(\begin{matrix}\lambda,\ \mu\\ \nu\end{matrix}\,;\, -zt^{n/s}\right) t^{np+n/s} \, \phi\left(t^{n/s}\right) dt$$

$$= (2\pi)^{\frac{1}{2}(2-n-s)} \, s^{\lambda+\mu+\nu-1} \, n^{k+m-np+1} \times$$

$$\int_0^\infty t^{n\rho-1}\, G^{s,\,2s+2n}_{2s+2n,\,2s+n}\left(\frac{z^s\, n^n}{t^n}\right|$$

$$\left.\begin{matrix}\Delta\,(n\,;\,n\rho),\ \Delta\,(n\,;\,n\rho-2m),\ \Delta\,(s\,;\,1-\lambda),\ \Delta\,(s\,;\,1-\mu)\\ \Delta\,(2n\,;\,2n\rho+2k-2m-1),\ \Delta\,(s\,;\,0),\ \Delta\,(s\,;\,1-\nu)\end{matrix}\right) \times f(t)\,dt.$$

On putting $t^{n/s}=u$ in the integral on left side, then multiplying both sides of the integeral by z and then replacin z by $\frac{1}{p}$ we get the result, (3.1)

Corollary 1.

If $\lambda=2m+1$, $\mu=1$, $\nu=m-k+3/2$, $\rho=0$, $k+m=\frac{1}{2}$ and $n=s=1$, then the theorem reduces to a known result due to Arya [1, p. 40].

Corollary 2.

If $n=s=1$ and $\mu=\nu$ the theorem reduces to a known theorem due to Saxena [6, p. 349].

Theorem 2.

If

$$f(t) \stackrel[\lambda]{S}{=} p^{n/s-\sigma+1}\,.\,\phi\,(p^{n/s}) \tag{3.4}$$

and

$$F\left\{\phi\,(t)\,;\,\begin{matrix}a,\,b\\ c\end{matrix}\right\} = \psi\,(p) \quad \ldots \tag{3.4}$$

then

$$\psi\,(p) = (2\pi)^{2-n-s}\; n^{\lambda}\; s^{a+b-c-1}\; p^{-1}\;\{\Gamma(\lambda)\}^{-1} \times$$

$$\int_0^{\infty} t^{\sigma-\lambda}\, G^{s+n,\, 2s+n}_{2s+n,\, 2s+n}\left(\frac{t^n}{p^s}\,\middle|\, \begin{matrix}\Delta(n\,;1-\sigma),\ \Delta(s\,;1-a),\ \Delta(s\,;1-b)\\ \Delta(n\,;\lambda-\sigma),\ \Delta(s\,;O),\ \Delta(s\,;1-c)\end{matrix}\right) f(t)\, dt. \tag{3.6}$$

provided the integral is convergent, n and s are positive integers, Stieltjes and Gauss hypergeometric transform of $|\,f(t)\,|$ and $|\,\phi(t)\,|$ exist respectively.

Proof : Applying the generalized Parseval-Goldstein theorem [6, p. 342] to relations (2.6) and (3.4), we have

$$\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\int_0^{\infty} t^{\sigma-2}\, {}_2F_1\left(\begin{matrix}a,\, b\\ c\end{matrix}\,;\,-zt^{n/s}\right) t^{n/s-\sigma+1}\,\phi(t^{n/s})\, dt$$

$$= (2\pi^{\,2-n-s(}\, n^{\lambda-1}\, s^{a+b-c}\,\{\Gamma(\lambda)\}^{-1} \times$$

$$\int_0^{\infty} t^{\sigma-\lambda}\, G^{s+n,\, 2s+n}_{2s+n,\, 2s+n}\left(z^s t^n\,\middle|\, \begin{matrix}\Delta(n\,;1-\sigma),\ \Delta(s\,;1-a),\ \Delta(s\,;1-b)\\ \Delta(n\,;\lambda-\sigma),\ \Delta(s\,;O),\ \Delta(s\,;1-c)\end{matrix}\right) f(t)\, dt.$$

Putting $t=u^{s/n}$ in the integral on left side, and multiplying both sides by z,

then replacing z by $\frac{1}{p}$ we get the result (3.6).

Corollary 1.

If $n=s=1$ and $b=c$ the theorm reduces to a known result due to Saxena [6, p. 343] by virtue of the identify due to Sharma, [7],

$$G^{22}_{22}\left(z\,\middle|\,\begin{matrix}1-a,\ 1-b\\ O,\ c-a-b\end{matrix}\right) = \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c-a)\,\Gamma\ c-b)}{\Gamma(c)}\, {}_2F_1\left(\begin{matrix}a,\, b\\ c\end{matrix}\,;\,1-z\right) \quad \ldots \tag{3.7}$$

Corollary 2.

If we put $a=2m+1$, $b=1$, $c=m-k+3/2$, $n=s=1$ and $\sigma=m-k+5/2$ we get a known result due to Arya, [1, p. 45].

Theorem 3.

If

$$F\left\{ f(t)\,;\, {\lambda, \mu \atop \nu} \right\} = \phi\left(p^{-n/s}\right) \quad \dots \qquad (3.8)$$

and

$$t^{s/n\,(\sigma-1)-m-\frac{1}{2}}\,\phi(t) \underset{k,\,m}{\overset{V}{=}} \psi(p\,;\,k,\,m)$$

then

$$\psi(p\,;\,k,\,m) = (2\pi)^{\frac{1}{2}(3-2n-s)}\, n^{\lambda+\mu-\nu}\, s^{k-\frac{1}{2}}\, p^{m+\frac{1}{2}} \times$$

$$\int_0^\infty t^{-\sigma}\, G^{2s+2n,\, n}_{s+2n,\, 2s+2n}\left(\frac{p^s}{t^n s^s} \,\middle|\, \begin{matrix} \Delta(n\,;\,1-\sigma),\ \Delta(n\,;\,\nu-\sigma),\ \Delta(s\,;\,1-k) \\ \Delta(n\,;\,\lambda-\sigma),\ \Delta(n\,;\,\mu-\sigma),\ \Delta(s\,;\,\frac{1}{2}\pm m) \end{matrix} \right) \times$$

$$f(t)\,dt. \quad \dots \quad \dots \qquad (3.9)$$

Provided the integral is convergent, Gauss, hypergeometric transform of $|f(t)|$ and Varma transform of $|\, t^{s/n\,(\sigma-1)-m-\frac{1}{2}}\,\phi(t)\,|$ exist, n and s are positive integers. $R(p) > 0$ and $R\{\sigma+n/s\,(\frac{1}{2}\pm m)\} > 0$.

Proof :

Using the relations (2.3) and (3.8) in the following property of Gauss-hypergeometric transform [7, p. 111].

If

$$F\left\{ f_1(t)\,;\, {\lambda, \mu \atop \nu} \right\} = \phi_1(p)$$

and

$$F\left\{ f_2(t)\,;\, {\lambda\ \mu \atop \nu} \right\} = \phi_2(p)$$

then

$$\int_0^\infty f_1(t)\,\phi_2\left(\frac{1}{t}\right) t^{-1}\,dt = \int_0^\infty f_2(t)\,\phi_1\left(\frac{1}{t}\right) t^{-1}\,dt \qquad \ldots \qquad (3.10)$$

we have,

$$\int_0^\infty t^{\sigma-2}\, e^{-\frac{1}{2}zt^{n/s}}\, W_{k,m}\left(zt^{n/s}\right) \Phi\left(t^{n/s}\right) dt = (2\pi)^{\frac{1}{2}(3-2n-s)}$$

$$n^{\lambda+\mu-\nu-1}\, s^{k+\frac{1}{2}} \times$$

$$\int_0^\infty t^{-\sigma}\, G^{2s+2n,\,n}_{s+2n,\,2s+2n}\left(\frac{z^s\,t^{-n}}{s^s}\,\middle|\,\begin{matrix}\Delta(n\,;\,1-\sigma),\ \Delta(n\,;\,\nu-\sigma),\ \Delta(s\,;\,1-k)\\ \Delta(n\,;\,\lambda-\sigma),\ \Delta(n\,;\,\mu-\sigma),\ \Delta(s\,;\,\frac{1}{2}\pm m)\end{matrix}\right) f(t)\,dt.$$

Putting $t^{n/s} = \mu$ in the integral on left side, we obtain,

$$\int_0^\infty e^{-\frac{1}{2}zu}\, W_{k,m}(zu)\, u^{s/n\,(\sigma-1}\,{}^{-1}\,\phi(u)\,du$$

$$= (2\pi)^{\frac{1}{2}(3-2n-s)}\, n^{\lambda+\mu-\nu}\quad s$$

$$\int_0^\infty t^{-\sigma}\, G^{2s+2n,\,n}_{s+2n,\,2s+2n}\left(\frac{z^s\,t^{-n}}{s^s}\,\middle|\,\begin{matrix}\Delta(n\,;\,1-\sigma),\ \Delta(n\,;\,\nu-\sigma),\ \Delta(s\,;\,1-k)\\ \Delta(n\,;\,\lambda-\sigma),\ \Delta(n\,;\,\mu-\sigma),\ \Delta(s\,;\,\frac{1}{2}\pm m)\end{matrix}\right) f(t)\,dt.$$

Hence on multiplying both sides by $z^{m+\frac{1}{2}}$ and after little simplification, we get the result (3.9) on replacing z by p.

Corollary : If $n=s=1$, $k+m=\frac{1}{2}$ and $\mu=\nu$ we obtain a known theorem [6, p. 349]

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ON INTEGRALS INVOLVING THE PRODUCT OF TWO HYPERGEOMETRIC SERIES AND SOME OPERATIONAL RESULTS IN TWO VARIABLES

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1. INTRODUCTION

The object of this paper is to establish two theorems which enable us to evaluate certain integrals involving the product of two Hypergeometric series. Some of the integrals are generalisations of integrals given in Watson's Bessel Functions and are believed to be new. Further we have evaluated certain operational images in two variables with the help of the integrals obtained by the same method as in case of the Hypergeometric integrals.

2. *Theorem* 1.

We have

$$
{}_pF_q\left(\begin{matrix} a_1, \ldots, a_p ; \\ b_1, \ldots, b_q ; \end{matrix} -yx\right) = \frac{\prod_{r=1}^{q} \Gamma(b_r)}{\prod_{r=1}^{p} \Gamma(a_r)} \cdot \frac{1}{2\pi i} \int_C \frac{\prod_{r=1}^{p} \Gamma(a_r+s)\Gamma(-s)}{\prod_{r=1}^{q} \Gamma(b_r+s)} (xy)^s \, ds \tag{1}
$$

where $|\arg x| < \pi$ and the poles of $\Gamma(a_r+s)$ lie to the left of the path on the temporary supposition that $R(a_r) > 0$, while the poles of $\Gamma(-s)$ lie to the right to the path. The relation (1) is first obtained for $|x| < 1$ and then since the integral on the right hand side of (1) represents a single valued analytic function of x in the domain defined by $|\arg x| < \pi$ and also the left hand side is an integral function of x, it follows by the general theory of analytic continuation that the relation (1) holds good for all x in the domain defined by $|\arg x| < \pi$.

Thus we have

$$
\int_0^\infty x^\lambda \; {}_pF_q\left(\begin{matrix} a_1, \ldots\ldots, a_p ; \\ a_1, \ldots\ldots, b_p ; \end{matrix} -yx\right) \; {}_lF_m\left(\begin{matrix} A_1, \ldots, A_l ; \\ B_1, \ldots, B_m ; \end{matrix} -zx\right) dx,
$$

$$
p = q,\ q-1 \text{ or } q+1 \ ; \ l = m,\ m-1 \text{ or } m+1 \ ; \ R(\lambda) > 1 \ ;
$$

$$= \frac{\prod_{r=1}^{q} \Gamma(b_r)}{\prod_{r=1}^{p} \Gamma(a_r)} \cdot \frac{1}{2\pi i} \int_C \frac{\prod_{r=1}^{p} \Gamma(a_r+s)\, \Gamma(-s)}{\prod_{r=1}^{q} \Gamma(b_r+s)}\, y^s ds \int_0^\infty x^{\lambda+s} \times$$

$$\times\ {}_lF_m \left(\begin{matrix} A_1, \ldots, A_l\, ; \\ B_1, \ldots, B_m\, ; \end{matrix} \quad -zx \right) dx$$

$$= \frac{\prod_{r=1}^{q} \Gamma(b_r) \prod_{r=1}^{m} \Gamma(B_r)}{\prod_{r=1}^{p} \Gamma(a_r) \prod_{r=1}^{l} \Gamma(A_r)}\, z^{-1-\lambda} \cdot \frac{1}{2\pi i}$$

$$\int_C \frac{\prod_{r=1}^{p} \Gamma(a_r+s) \prod_{r=1}^{l} \Gamma(A_r-\lambda-1-s)\, \Gamma(-s)\, \Gamma(1+\lambda+s)}{\prod_{r=1}^{q} \Gamma(b_r+s) \prod_{r=1}^{m} \Gamma(B_r-\lambda-1-s)} \times \left(\frac{y}{z}\right)^s ds$$

$$\text{Or} \int_0^\infty x^\lambda\, {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p\, ; \\ b_1, \ldots, b_q\, ; \end{matrix} \ -yx \right) {}_lF_m \left(\begin{matrix} A_1, \ldots, A_l\, ; \\ B_1, \ldots, B_m\, ; \end{matrix} \ -zx \right) dx$$

$$= \frac{\prod_{r=1}^{q} \Gamma(b_r) \prod_{r=1}^{m} \Gamma(B_r)}{\prod_{r=1}^{p} \Gamma(a_r) \prod_{r=1}^{l} (A_r)}\, z^{-1-\lambda}\, G^{l+1,\, p+1}_{p+m+1,\, q+l+1}$$

$$\left(\frac{y}{z} \middle| \begin{matrix} 1-a_1, \ldots, 1-a_p\, , -\lambda, B_1-\lambda-1, \ldots, B_m-\lambda-1 \\ 0, A_1-\lambda-1, \ldots, A_l-\lambda-1,\ 1-b_1, \ldots, 1-b_q \end{matrix} \right) \tag{2}$$

under the following conditions of convergence*.

For the change in the order of integrations :

$p=q,\ q+1\ ;\ l=m,\ m+1\ :\ R\,(\lambda+c) > -1,\ R\,(-\lambda-c+A_r) > 1\ ;\ (r=1, \ldots, l)\ (i)$

$p=q-1\ ;\ l=m,\ m+1\ ;$ In addition to $(i)\ \frac{1}{2}\sum_1^q b_r-\frac{1}{2}\sum_1^p a+c>1$

$p=q,\ q+1\ ;\ l=m-1\ ;\ R(\lambda+c) > -1,\ R(-\lambda-c-\frac{1}{2}\sum_1^l A_r+\frac{1}{2}\sum_1^m B_r) > 5/4.$

$p=q-1,\ l=m-1;\ R(\lambda+c)>-1,\ R(\frac{1}{2}\sum_1^q p_r-\frac{1}{2}\sum_1^p a_r+c, >1,\ R(-\lambda-c-\frac{1}{2}\sum_1^l A_r+\frac{1}{2}\sum_1^m B_r) > 5/4$

For the convergence of the Integral (2) :

$q=p,\ p-1\ ;\ m=l,\ l-1\ :\ R\,(-\lambda+a_r+A_s) > 1\ ;\ (r=1, \ldots, p\ ;\ s=1, \ldots, l)$

---

* For the change in the order of integration we use Fubni's Theorem [7].

Also we note that $\Gamma\left(\frac{s}{2}\right) \sim e^{-\pi/4\,|t|}\left|\frac{t}{2}\right|^{c-1/2}$, $s=c+it$ and

$${}_pF_{p+m-1}\left[\begin{matrix}b_1, \ldots, b_p\ ; \\ a_1, \ldots, a_{p+m-1}\ ;\end{matrix}\ \mp\left(\frac{x}{m}\right)^m\right] = O\ (1) \text{ for small } x$$

$$\sim x^{\theta}\ exp\left[\ x\, e^{i\,(\frac{1}{2}\pm\frac{1}{2})\,\pi/m}\ \right] + \sum_{r=1}^{p} a_r\, x^{-mb_r} \text{ for large } x$$

where $\theta = \sum_{r=1}^{p} b_r - \sum_{r=1}^{p+m-1} a_r + \frac{1}{2}(m-1)$

$q=p,\ p-1\ ;\ m=l+1 : R\left(-\lambda+a_r-\frac{1}{2}\sum_1^l A_s+\frac{1}{2}\sum_1^m B_s\right) > 5/4,\ (r=1, \ldots, p)$

$q=p+1\ ;\ m=l-1,\ l : R\left(-\lambda-\frac{1}{2}\sum_1^p a_r+\frac{1}{2}\sum_1^q b_r+A_s\right) > 5/4\ ;\ (s=1, \ldots, l)$

$$R\left(-\lambda+a_r+A_s\right) > 1,\ (r=1, \ldots, p\ ;\ s=1, \ldots, l)$$

$q=p+1\ ;\ m=l+1 : R\left(-\lambda-\frac{1}{2}\sum_1^q a_r+\frac{1}{2}\sum_1^q b_r-\frac{1}{2}\sum_1^l A_r+\frac{1}{2}\sum_1^m B_r\right) > 3/2$

$$R\left(-\lambda-\frac{1}{2}\sum_1^p a_r+\frac{1}{2}\sum_1^q b_r+A_s\right) > 5/4,\ (s=1, \ldots, l)$$

$$R\left(-\lambda-\frac{1}{2}\sum_1^l A_r+\frac{1}{2}\sum_1^m B_r+a_s\right) > 5/4,\ (s=1, \ldots, p)$$

$$R\left(-\lambda+a_r+A_s\right) > 1,\ (r=1, \ldots, p\ ;\ s=1, \ldots, l)$$

Theorem 2.

Similarly as in theorem 1, we have

$$\int_0^\infty x^\lambda\ {}_pF_q\left(\begin{matrix}a_1, \ldots, a_p\ ; \\ b_1, \ldots, b_q\ ;\end{matrix}\ -yx\right){}_lF_m\left(\begin{matrix}A_1, \ldots, A_l\ ; \\ B_1, \ldots, B_m\ ;\end{matrix}\ -z/x\right) dx, \qquad (3)$$

$$p=q,\ q-1 \text{ or } q+1\ ;\ l=m,\ m-1 \text{ or } m+1\ ;$$

$$= \frac{\prod_{r=1}^{q}\Gamma(b_r)}{\prod_{r=1}^{p}\Gamma(a_r)} \cdot \frac{1}{2\pi i}\int_C \frac{\prod_{r=1}^{p}\Gamma(a_r+s)\,\Gamma(-s)}{\prod_{r=1}^{q}\Gamma(b_r+s)} y^s\, ds$$

$$\int_0^\infty x^{\lambda+s}\ {}_lF_m\left(\begin{matrix}A_1, \ldots, A_l\ ; \\ B_1, \ldots, B_m\ ;\end{matrix}\ -z/x\right) dx$$

$$= \frac{\prod_{r=1}^{q} \Gamma(b_r) \prod_{r=1}^{m} \Gamma(B_r)}{\prod_{r=1}^{q} \Gamma(a_r) \prod_{r=1}^{l} \Gamma(A_r)} z^{\lambda+1} \cdot \frac{1}{2\pi i}$$

$$\int_C \frac{\prod_{r=1}^{p} \Gamma(a_r+s)\, \Gamma(-s)\, \Gamma(-\lambda-1-s) \prod_{r=1}^{l} \Gamma(A_r+\lambda+s+1)}{\prod_{r=1}^{q} \Gamma(b_r+s) \prod_{r=1}^{m} \Gamma(B_r+\lambda+1+s)} (yz)^s ds$$

$$= \frac{\prod_{r=1}^{q} \Gamma(b_r) \prod_{r=1}^{m} \Gamma(B_r)}{\prod_{r=1}^{p} \Gamma(a_r) \prod_{r=1}^{l} \Gamma(A_r)} z^{\lambda+1} G^{2,\, p+l}_{p+l,\, q+m+2}$$

$$\left(yz \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p, -A_1-\lambda, \ldots, -A_l-\lambda \\ 0, -1-\lambda, 1-b_1, \ldots, 1-b_q, -B_1-\lambda, \ldots, -B_m-\lambda \end{matrix} \right) \qquad (3')$$

under the following conditions of convergence.

For the change in the order of integratione ;

$p=q, q+1$; $l=m, m+1$; $R(\lambda+c+A_r)>-1$, $(r=1, \ldots, l)$ (*i*) $R(\lambda+e)<-1$ (*ii*)

$p=q+1, q$; $l=m-1$: $R(\lambda+c-\frac{1}{2}\sum_1^l A_r+\frac{1}{2}\sum_1^m B_r)>-\frac{3}{4}$ (*iii*) $R(\lambda+c)<-1$

$p=q-1$, $l=m-1$; $R(\frac{1}{2}\sum_1^q b_r-\frac{1}{2}\sum_1^p a_r+c)>1$ (*iv*) and (*ii*) and (*iii*)

$p=q-1$, $l=m, m+1$; (*i*), (*ii*) and (*iv*)

For the convergence of the integral (3) :

At the origin

$q = p \; p-1, p+1; \; l = m, \; m+1 : R(\lambda + A_r) > -1, \; (r = 1, \; .., \; l)$

$q = p, \; p-1, p+1 \; ; \; l = m-1 : R(\lambda - \frac{1}{2} \sum_1^l A_r + \frac{1}{2} \sum_1^m B_r) > -3/4$

At infinity

$l = m-1, \; m, \; m+1 \; ; \; q = p-1, \; p : R(-\lambda + a_r) > 1, \; (r = 1, \; .., \; p)$

$l = m-1, \; m, \; m+1 \; ; \; q = p+1 : R(-\lambda - \frac{1}{2} \sum_1^p a_r + \frac{1}{2} \sum_1^q b_r) > 5/4$

3. *Example* :

$$(1) \int_0^\infty x^\lambda \; {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p ; \\ b_1, \ldots, b_q ; \end{matrix} -yx\right) {}_0F_1 \,(B \; ; \; -zx) \; dx = \frac{\Gamma(B) \; \Gamma(\lambda+1)}{\Gamma(B-\lambda-1) z^{\lambda+1}}$$

$${}_{p+2}F_q \left(\begin{matrix} a_1, \ldots, a_p, \lambda+1, \; 2+\lambda - B \; ; \\ b_1, \ldots, b_q \; ; \end{matrix} \frac{y}{z}\right)$$

$$\text{or } {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p; \\ b_1 \ldots, b_q \; ; \end{matrix} y\right) = \frac{\Gamma(B-\lambda-1)}{\Gamma(B) \; \Gamma(\lambda+1)} \int_0^\infty x^\lambda$$

$${}_pF_{q+2} \left(\begin{matrix} a_1, \ldots, a_p \; ; \\ b_1, \ldots, b_q \; ; \lambda+1, \; 2+\lambda-B, \end{matrix} -yx\right) {}_0F_1(B \; ; \; -x) dx,$$

$p = q+1$, if $p \geqslant q+2$ one of the a_r's must be a negative integer and

$R(\lambda) > -1$;

under the usual conditions of convergence.

The gives the integral representation of a Hypergeometric Series, viz,

$$ {}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} ; \\ b_1, \ldots, b_q ; \end{matrix} y\right) = \frac{\Gamma(B-\lambda-1)}{\Gamma(B)\ \Gamma(\lambda+1)} \int_0^\infty x^\lambda \ {}_{q+1}F_{q+2} $$

$$ \left(\begin{matrix} a_1, \ldots, a_{q+1} ; \\ b_1, \ldots, b_q, \lambda+1, 2+\lambda-B ; \end{matrix} -yx\right) {}_0F_1(B-x)dx $$

where $R(\lambda) > -1$, $R(\Sigma a_r - \Sigma b_r) < 0$, $R(a_r + B/2 - \lambda) > 5/4$, $R(B) > 0$.

As a particular case we have

$$ {}_3F_2\left(\begin{matrix} a, b, c ; \\ d, e ; \end{matrix} y\right) = \frac{\Gamma(B-\lambda-1)}{\Gamma(B)\ \Gamma(\lambda+1)} \int_0^\infty x^\lambda $$

$$ {}_3F_4\left(\begin{matrix} a, b, c ; \\ d, e, \lambda+1, 2+\lambda-B ; \end{matrix} -yx\right) {}_0F_1(B ; -x)\, dx. $$

$R(\lambda) > -1$, $R(-a-b-c+d+e) > 0$, $R(a, b, c, -\lambda+B/2) > 5/4$.

$$ (2) \left\{ \int_0^\infty x^\lambda \ {}_2F_1\left(\begin{matrix} a, b ; \\ \lambda+1 ; \end{matrix} -px\right) {}_2F_3\left(\begin{matrix} A, A-\frac{1}{2} ; \\ 2A+b-\lambda-2, 2A+a-\lambda-2, \lambda+1 ; \end{matrix} -qx\right) dx \right. $$

$$ = \frac{2\,\Gamma(\frac{1}{2})[\Gamma(\lambda+1)]^2\ \Gamma(2A+b-\lambda-2)\ \Gamma(2A+a-\lambda-2)\ q^{1-A}\ p^{A-\lambda-2}}{\Gamma(A)\ \Gamma(A-\frac{1}{2})\ \Gamma(a)\ \Gamma(b)} $$

$$ I_{2A+a+b-2\lambda-4}(\sqrt{q/p})\ K_{b-a}(\sqrt{q/p}). $$

$R(\lambda) > -1$, $R(a, b, +A-\lambda-3/2) > 0$, $R(5-3b-2A-a+3\lambda) < 0$.

$$ (3) \left\{ \int_0^\infty x^\lambda \ {}_1F_2\left(\begin{matrix} a ; \\ b, 1+\lambda ; \end{matrix} -px\right) {}_2F_1\left(\begin{matrix} 3/2+\lambda, b+\lambda ; \\ 1+a+\lambda ; \end{matrix} -qx\right) dx \right. $$

$$ = \frac{\Gamma(\frac{1}{2})\ \Gamma(b)\ \Gamma(1+\lambda)\ \Gamma(1+a+\lambda)\ \Gamma(a+b-1)}{p^{\frac{1}{2}}\ q^{\lambda+\frac{1}{2}}\ \Gamma(a)\ \Gamma(3/2+\lambda)\ \Gamma(b+\lambda)\ \Gamma(2-2a)} W_{\frac{1}{2}-a,\, b-1}(2\sqrt{p/q}) M_{a-\frac{1}{2},\, b-1}(2\sqrt{p/q}), $$

$R(\lambda) > -1$, $R(a/2-b/2-\lambda/2-3/4) < 0$, $R(a/2-3b/2-\lambda/2+3/4) < 0$, $R(a+\frac{1}{2}) > 0$, $R(a+b-1) > 0$.

This integral is a generalisation of an integral given in Watson's Bessel Functions, p. 385 (2), when $u=v-1$.

$$(4)\quad \int_0^\infty x^\lambda\, {}_1F_2\begin{pmatrix} a\,; & \\ b,\ \lambda+1\,; & -px\end{pmatrix} {}_2F_1\begin{pmatrix} 3/2+\lambda-a,\ 2+\lambda-a\,; & \\ \lambda+1\,; & -qx\end{pmatrix} dx$$

$$= \frac{\pi\,\Gamma(b)\,[\,\Gamma(\lambda+1)\,]^2\, p^{3/4-a/2-b/2}\, q^{-7/4+a/2+b/2-\lambda}}{\Gamma(a)\ \Gamma(2+\lambda-a)\ \Gamma(\lambda-a+3/2)} \Big[\, I_{b-a-\frac{1}{2}}(2\sqrt{p/q}) - L_{b-a-\frac{1}{2}}(2\sqrt{p/q}) \Big],$$

$R(\lambda) > -1$, $R(3a/2-b/2-\lambda/2-3/4) < 0$.

$$(5)\quad \int_0^\infty x^\lambda\, {}_1F_1(a\,;\lambda+1\,;-px)\, {}_2F_1\begin{pmatrix} A,\ B\,; & \\ \lambda+1\,; & -qx\end{pmatrix} dx$$

$$= \frac{[\Gamma(\lambda+1)]^2\ \Gamma(A+a-\lambda-1)\ \ \Gamma(B+a-\lambda-1)}{\Gamma(a)\ \Gamma(A)\ \Gamma(B)}\, e^{p/(2q)}\, p^{(A+B-2\lambda-3)/2}$$

$$q^{(1-A-B)/2}\ W_{\lambda-a-A/2-B/2+3/2,\ A-B/2}\ (p/q),$$

$R(\lambda) > -1$, $R(\lambda-a-A,\ B,\ +1) < 0$.

If we put $B=\lambda+1=a$, we get (22, p. 139) of Integral Transform Vol. 1.

Vide Modern Analysis, third edition, p. 340.

$$(6)\quad \int_0^\infty x^{-3/2}\ {}_0F_1(b\,;-yx)\ {}_0F_1(b\,;-z/x)\, dx = [\,\Gamma(b)\,]^2\, z^{-b/2}\, y^{1-b/2}\, J_{2b-2}(4\sqrt[4]{yx}),$$

$R(b-\frac{1}{2}) > 0$.

$$(7)\quad \int_0^{\infty} x^{a-A-2/2}\ {}_1F_1\left(a\,;\,a+A-\tfrac{1}{2})\,;\,-yx\right)\ {}_1F_1\left(A\,;\,a+A-\tfrac{1}{2}\,;\,-z/x\right)\,dx$$

$$=\frac{2\,[\,\Gamma\,(a+A-\tfrac{1}{2})]^2\ \Gamma(\tfrac{1}{2})}{\Gamma\,(a)\ \Gamma(A)}\, z^{\frac{1}{2}-A}\ y^{1-a}\ I_{a+A-3/2}\,(\sqrt{yz}\,)\ K_{a-A-\frac{1}{2}}\,(\sqrt{yz}\,),$$

$R(a) > \tfrac{1}{2}$, $R\,(A) > -\tfrac{1}{2}$,

$$(8)\int_0^{\infty} x^{\lambda}\ {}_1F_1\,(A\,;\,\lambda+1\,;\,-yx)\ {}_pF_q\begin{pmatrix} a_1,\,\ldots,\,a_p\,; & -zx \\ b_1,\,\ldots,\,b_q\,; & \end{pmatrix} dx$$

$$=\frac{\prod_{r=1}^{q}\Gamma\,(b_r)\ \prod_{r=1}^{p}\Gamma\,(a_r-\lambda-1+A)\ \ \Gamma(\lambda+1)\ z^{-\lambda-1+A}\,y^{-A}}{\prod_{r=1}^{p}\Gamma(a_r)\ \prod_{r=1}^{q}\Gamma\,(b_r)-\lambda-1+A)}$$

$${}_{p+1}F_q\begin{pmatrix} A,\,a_1-\lambda-1+A,\,\ldots,\,a_p-\lambda-1+A\,; & -\dfrac{z}{y} \\ b_1-\lambda-1+A,\,\ldots,\,b_q-\lambda-1+A\,; & \end{pmatrix}.$$

under the usual conditions of convergence.

$$(9)\int_0^{\infty} x^{\lambda}\ {}_1F_2\begin{pmatrix} A\,; & -yx \\ B,\,\lambda+1\,; & \end{pmatrix}\ {}_pF_q\begin{pmatrix} a_1,\,\ldots,\,a_p\,; & -zx \\ b_1,\,\ldots,\,b_q\,; & \end{pmatrix} dx$$

$$=\frac{\Gamma(\lambda+1)\ \prod_{r=1}^{q}\Gamma(b_r)\ \prod_{r=1}^{p}\Gamma(a_r-\lambda-1+A)\ \Gamma(B)}{\prod_{r=1}^{p}\Gamma\,(a_r)\ \prod_{r=1}^{q}\Gamma\,(b_r-\lambda-1+A)\ y^{A}\,.z^{\lambda+1-A}}$$

$${}_{p+2}F_q\begin{pmatrix} A,\,1+A-B,\,a_1-\lambda-1+A,\,\ldots,\,a_p-\lambda+1+A\,; & z/y \\ b_1-\lambda-1+A,\,\ldots,\,b_q-\lambda-1+A\,; & \end{pmatrix},$$

under the usual conditions of convergence. This integral is the generalisation of the integral given in Watson's Bessel Functions, for particular values of p and q (3, 4, p. 404).

$$(10) \int_0^{\infty} x^{\lambda} \; {}_0F_1(b\,;\,-x)\; {}_0F_1(B;\,-x)\,dx = \frac{\Gamma(B)\;\Gamma(1+\lambda)\;\Gamma(b)\;\Gamma(b+B-2\lambda-3)}{\Gamma(B-\lambda-1)\;\Gamma(b-\lambda-1)\;\Gamma(b+B-\lambda-2)},$$

$R(\lambda)>-1,\; R(b+B-2\lambda)>3.$

This is a known result.

$$(11) \int_0^{\infty} x^{\lambda} \; {}_1F_2\begin{pmatrix} a\,; & \\ a+B-\lambda-1,\; a-\lambda\,; & -x \end{pmatrix} {}_0F_1\;(B\,;\,-x)\,dx$$

$$= \frac{\Gamma(B)\;\Gamma(1+\lambda)\;\Gamma(1+a/2)\;\Gamma(a+B-\lambda-1)\;\Gamma(a/2+B-2\lambda-2)\;\Gamma(a-\lambda)}{\Gamma(B-\lambda-1)\;\Gamma(1+a)\;\Gamma(a/2-\lambda)\;\Gamma(a/2+B-\lambda-1)\;\Gamma(a+B-2\lambda-2)};$$

$R(\lambda) > -1,\; R(a+2B-4\lambda) > 4,\; R(2a+B-2\lambda) > 5/2;$

$$(12) \int_0^{\infty} x^{\lambda} \; {}_3F_4\begin{pmatrix} a,\; 1+a/2,\; c\,; & \\ a/2,\; 1+a-c,\; a+B-\lambda-1,\; a-\lambda\,; & -x \end{pmatrix} {}_0F_1\;(B\,;\,-x)\;dx.$$

$$= \frac{\Gamma(B)\;\Gamma(\lambda+1)\;\Gamma(1+a-c)\;\Gamma(a+B-\lambda-1)\;\Gamma(a-\lambda)\;\Gamma(a+B-c-2\lambda-2)}{\Gamma(B-\lambda-1)\;\Gamma(a+1)\;\Gamma(a+c+\lambda)\;\Gamma(a+B-2\lambda-2)\;\Gamma(a-c+B-\lambda-1)};$$

$R(\lambda) > -1;\; R(1+a/2,\, a,\, c,\, +B/2-\lambda) > 5/4,\; R(c-a-B+2\lambda+2)<0.$

This is a generalisation of (11). When we put $c=a/2$, we fall back upon (11).

$$(13) \int_0^{\infty} x^{\lambda} \; {}_1F_2\begin{pmatrix} a\,; & \\ 3/2+\lambda/2+a/2-B/2,\; 2+2\lambda\,; & -x \end{pmatrix} {}_0F_1\;(B\,;\,-x)\,dx$$

$$= \frac{\Gamma(B)\;\Gamma(\lambda+1)\;\Gamma(\tfrac{1}{2})\;\Gamma(\lambda+3/2)\;\Gamma(3/2+\lambda/2+a/2-B/2)\;\Gamma(\lambda/2-a/2+B/2+\tfrac{1}{2})}{\Gamma(B-\lambda-1)\;\Gamma(a/2+\tfrac{1}{2})\;\Gamma(3/2+\lambda/2-B/2)\;\Gamma(\lambda-a/2+3/2)\;\Gamma(\lambda/2+B/2+\tfrac{1}{2})},$$

$R(\lambda) > -1,\; R(\lambda-B/2-a) < -5/4,\; R(a-\lambda-B-1)<0.$

$$(14) \int_0^{\infty} x^{\lambda}\; {}_1F_1\,(a\,;\,\lambda+1\,;\,-x)\; {}_3F_3\begin{pmatrix} 2+\lambda & a/2,\; B,\; C\,; & \\ 1+\lambda & a/2,\; 3+2\lambda-B-a,\, 3+2\lambda-c-a\,; & x \end{pmatrix} dx$$

$$=\frac{\Gamma(\lambda+1)\,\Gamma(1+\lambda-a/2)\,\Gamma(3+2\lambda-B-a)}{\Gamma(2+\lambda-a/2)\,\Gamma(B)\,\Gamma(C)\,\Gamma(a/2)\,\Gamma(1+a)}$$

$$\frac{\Gamma(3+2\lambda-c-a)\,\Gamma(1+a/2)\,\Gamma(B+a-\lambda-1)\,\Gamma(c+a-\lambda-1)}{\Gamma(3+2\lambda-B-C-a)}$$

$R(\lambda)>-1$, $R(B, C, +a-\lambda)>1$, $R(a)>-2$.

$$(15)\quad \int_0^\infty x^\lambda \; {}_1F_1(a\,;\lambda+1\,;-x)$$

$${}_4F_4\left(\begin{matrix} A_1,\ A_2,\ A_3,\dfrac{A_1-a+\lambda+3}{2}\,; \\ \lambda+1,\ 2+\lambda+A_1-A_2-a,\ 2+\lambda+A_1-A_3-a,\ \dfrac{A_1-a+\lambda+1}{2}\,; \end{matrix}\ -x\right)dx$$

$$\frac{[\Gamma(\lambda+1)]^2\,\Gamma\left(\dfrac{A_1-a+\lambda+1}{2}\right)\Gamma(2+\lambda+A_1-A_2-a)\,\Gamma(2+\lambda+A_1-A_3-a)}{\Gamma(a)\,\Gamma(A_1)\,\Gamma\left(\dfrac{A_1-a+\lambda+3}{2}\right)\Gamma(A_2)\,\Gamma(A_3)}$$

$$\frac{\Gamma(A_1+a-\lambda-1)\,\Gamma\left(\dfrac{A_1+a-\gamma+1}{2}\right)\Gamma(A_2+a-\lambda-1)}{\Gamma\left(\dfrac{A_1+a-\lambda-1}{2}\right)\Gamma(A_1+a-\lambda)}\times$$

$\times\dfrac{\Gamma(A_3+a-\lambda-1)}{\Gamma(A_1-A_2-A_3-a+\lambda+2)}$, $R(\lambda)>-1$, $R(A_1, A_2, A_3, +a-\lambda-1)>0$,

$R(a+A_1-\lambda+1)>0$.

$$(16)\quad \int_0^\infty x^\lambda \; {}_1F_2\left(\begin{matrix} a\,; \\ b,\ \lambda+1\,; \end{matrix} -x\right) {}_1F_2\left(\begin{matrix} A\,; \\ 3+2\lambda-A-a,\ 1+\lambda+b-a\,; \end{matrix} -x\right) dx.$$

$$= \frac{\Gamma(\lambda+1)\,\Gamma(3+2\lambda-A-a)\,\Gamma(1+\lambda+b-a)\,\Gamma(A+a-\lambda-1)\,\Gamma(1+a/2)\,\Gamma(b)}{\Gamma(A)\,\Gamma(b-a)\,\Gamma(1+a)\,\Gamma(b-a/2)\,\Gamma(2+\lambda-a/2-A)}$$

$$\frac{\Gamma(1+\lambda+b-A-3a/2)}{\Gamma(1+\lambda+b-A-a)},$$

$R(\lambda)>-1$, $R(A+a-\lambda)>1$, $R(2\lambda-2A+2b-3a+2)>0$, $R(b-a-\lambda+2A)>-3/2$,
$R(b+\lambda-2A)>-3/2$.

$$(17)\quad \int_0^\infty x^\lambda \; {}_1F_2\left(\begin{matrix} a\,; \\ b,\ 1+\lambda\,; \end{matrix} -x\right) {}_1F_2\left(\begin{matrix} A\,; \\ B,\ \lambda+1\,; \end{matrix} -x\right) dx.$$

$$\frac{[\Gamma(\lambda+1)]^2\,\Gamma(b)\,\Gamma(B)\,\Gamma(A+a-\lambda-1)\,\Gamma(B-A-a-1+b)}{\Gamma(A)\,\Gamma(b-a)\,\Gamma(a)\,\Gamma(B-A)\,\Gamma(B-\lambda-2+b)},$$

$R(\lambda)>-1$, $R(B-A-a+b)>1$, $R(2A-\lambda-a+b)>3/2$, $R(B-A-\lambda+2a)>-3/2$,
$R(a+A-\lambda)>1$.

This is a generalisation of 3, 4, p. 404 in Bessel Functions, when $a=b$.

$$(18)\quad \int_0^\infty x^\lambda \; {}_1F_2\left(\begin{matrix} a\,; \\ b,\ \lambda+1\,; \end{matrix} -x\right) {}_2F_3\left(\begin{matrix} A,\ B\,; \\ 2+\lambda+A-B-a,\ A+b-a,\ \lambda+1\,; \end{matrix} -x\right) dx.$$

$$= \frac{[\Gamma(\lambda+1)]^2\,\Gamma(b)\,\Gamma(2+\lambda+A-B-a)\,\Gamma(A+b-a)\,\Gamma(A+a-\lambda-1)\,\Gamma(B+a-\lambda-1)}{\Gamma(a)\,\Gamma(A)\,\Gamma(B)\,\Gamma(b-a)\,\Gamma(A+a-\lambda)\,\Gamma(A+b-B-a)}$$

$$\frac{\Gamma\left(\dfrac{1+A+a-\lambda}{2}\right)\Gamma\left(\dfrac{A+\lambda-3a-2B+1+2b}{2}\right)}{\Gamma\left(\dfrac{A-a-\lambda-1+2b}{2}\right)\Gamma\left(\dfrac{A+\lambda-a-2B+3}{2}\right)}$$

$R(\lambda) > -1$, $R[2(A, B) - a + b - \lambda] > 3/2$, $R(A, B + a - \lambda) > 1$, $R(-b/2 - A/2 - 1/4) < 0$,
$R(2B + 3a - \lambda - A - 2b - 1) < 0$.

$$(19) \quad \int_0^\infty x^\lambda \, {}_1F_2\left(\begin{matrix} a\,; \\ b,\ \lambda+1\,; \end{matrix} -x\right)$$

$${}_3F_4\left(\begin{matrix} 2+\lambda-a/2,\ B,\ D\,; \\ 1+\lambda-a/2,\ 3+2\lambda-a-B,\ 1+\lambda+b-a,\ 3+2\lambda-a-D\,; \end{matrix} -x\right) dx.$$

$$= \frac{\Gamma(\lambda+1)\,\Gamma(b)\,\Gamma(1+\lambda-a/2)\,\Gamma(3+2\lambda-a-B)\,\Gamma(3+2\lambda-a-D)\,\Gamma(1+\lambda+b-a)}{\Gamma(B)\,\Gamma(D)\,\Gamma(2+\lambda-a/2)\,\Gamma(b-a)\,\Gamma(a/2)\,\Gamma(1+a)\,\Gamma(3+2\lambda-a-B-D)}$$

$$\frac{\Gamma(1+a/2)\,\Gamma(B-\lambda-1+a)}{\Gamma(1+\lambda+b-a-B)} \times$$

$$\times \frac{\Gamma(D-\lambda-1+a)\,\Gamma(2+2\lambda-2a-B-D+b)}{\Gamma(1+\lambda+b-a-D)}, \ R(\lambda) > -1\,;\ R(a) > -2,$$

$R(\lambda+b-2a) > -5/2$,

$R(a+B, D, -\lambda) > 1$, $R(2B, 2D, -a+b-\lambda) > 3/2$, $R(b-a+3\lambda-2B-2D) > -7/2$,
$R(b-2a-B-D+2\lambda) > -2$.

$$(20) \quad \int_0^\infty x^\lambda \, {}_1F_2\left(\begin{matrix} a\,; \\ b,\ \lambda+1\,; \end{matrix} -x\right) {}_1F_2\left(\begin{matrix} A\,; \\ 2+\lambda-b/2,\ 2A+a-\lambda-1\,; \end{matrix} -x\right) dx$$

$$= \frac{\Gamma(\lambda+1)\,\Gamma(b)\,\Gamma(2+\lambda-b/2)\,\Gamma(2A+a-\lambda-1)\,\Gamma(A+a-\lambda-1)\,\Gamma(\tfrac{1}{2})}{\Gamma(A)\,\Gamma(b-a)\,\Gamma(2A+2a-2\lambda-2)\,\Gamma\left(\frac{a+1}{2}\right)\,\Gamma\left(\frac{2+a-b}{2}\right)}$$

$$\frac{\Gamma(A+a-\lambda-\tfrac{1}{2})\,\Gamma(A-\lambda-1+b/2)}{\Gamma\left(A-\lambda+\frac{a-1}{2}\right)\,\Gamma\left(A-\lambda-1+\frac{a+b}{2}\right)}$$

$R(\lambda) > -1$, $R(a+A-\lambda) > 1$, $R(a/2-A-b/2+\lambda/2+3/4) < o$, $R\left(\frac{1+\lambda-A}{2}-b/4\right) < o$,

$R(b/4-3a/2-A/2+\lambda) < -3/4$.

$$(21) \quad \int_0^\infty x^\lambda \; {}_1F_2\left(\begin{matrix}a\,; \\ b,\; 1+\lambda\,;\end{matrix}\; -x\right) {}_2F_3\left(\begin{matrix}A,\; B\,; \\ \lambda+1,\; 3+a+\lambda-2b,\; \frac{A+B+1}{2}\,;\end{matrix}\; x\right) dx$$

$$= \frac{[\Gamma(\lambda+1)]^2\, \Gamma(b)\, \Gamma(3+a+\lambda-2b)\, \Gamma\left(\frac{A+B+1}{2}\right) \Gamma(a-b+3/2)}{\Gamma(a)\, \Gamma(A)\, \Gamma(B)\, \Gamma\left(\frac{A+a-\lambda}{2}\right) \Gamma\left(\frac{B-\lambda+a}{2}\right) \Gamma(b-a)\, \Gamma\left(2-b+\frac{\lambda+a-A}{2}\right)}$$

$$\frac{\Gamma\left(\lambda-b+\frac{5-A-B}{2}\right) \Gamma(A+a-\lambda-1)\, \Gamma(B+a-\lambda-1)\, \Gamma(\tfrac{1}{2})}{\Gamma\left(2-b+\frac{a+\lambda-B}{2}\right) \Gamma(2+2a-2b)}$$

$R(\lambda) > -1$, $R(A, B, -\lambda+a-1) > o$, $R\left(\frac{A+B}{4}+b-\frac{3a}{2}-1\right) < o$,

$R(A, B, +b/2-\lambda/2-a/2-3/4) > o$,

$R\left(\frac{A+B-5}{4}+\frac{b-\lambda}{2}\right) < o$.

4. Some Operational Images in two variables ** :

** Here we have $\phi(p, q) = pq \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\, dx\, dy$. For the coditions under which this result holds, please see [2, 4, 5] and also Operational Calculus by Vonder Pol and Bremmer.

(1) We have $\int_0^\infty t^{\lambda}\ e^{-t^2/2}\ .\ e^{-pt^2}\,dt = \dfrac{\Gamma\left(\frac{\lambda+1}{2}\right) 2^{(\lambda-1)/2}}{(1+2p)^{(\lambda+1)/2}}$, $R(\lambda) > -1$ (a)

Writing $(pq)^{-1/2}$ for p and multiplying both the sides by $p^{-1/2}\,(pq)^{1-v}$ and then interpreting we get

$$(4xy)^{(2v-1)/4}\,(\pi y)^{-\frac{1}{2}} \int_0^\infty t^{-2v+1}\, e^{-t^2/2}\, J_{2v-1}\left[(64\,xy)^{\frac{1}{4}}\, t\right] d$$

$$\doteqdot \frac{\Gamma\left(\frac{\lambda+1}{2}\right) 2^{(\lambda-1)/2}\ p^{-\frac{1}{2}}\,(pq)^{1-v}}{\left(1+\frac{2}{\sqrt{pq}}\right)^{(\lambda+1)/2}}$$

or $(xy)^{v-1/2}\ y^{-\frac{1}{2}}\ e^{-2\,(xy)^{\frac{1}{2}}}\ M_{\lambda+1/2-v,\ v-\frac{1}{2}}\,(4\sqrt{xy}) \doteqdot$

$$2\,\Gamma(2v)\ \Gamma(\tfrac{1}{2})\,p^{-\frac{1}{2}}\,(pq)^{1-v}\left(1+\frac{2}{\sqrt{pq}}\right)^{-(\lambda+1)/2},$$

$R(v) > 0.$ (1)

Putting $\lambda=1$, $v=1/2$, we get

$$y^{-\frac{1}{2}}\exp(-4\sqrt{xy}) \doteqdot q\ p^{\frac{1}{2}}\ \Gamma(\tfrac{1}{2})\ (2+\sqrt{pq})^{-1} \qquad (2)$$

Again in (a) writing $\dfrac{1}{p}+\dfrac{1}{q}$ for p and multiplying both the sides by $p^{-m}\,q^{-n}$ and then interpreting we get

$$\int_0^\infty t^\lambda \; e^{-t^2/2} \left(\frac{x}{t^2}\right)^{m/2} \left(\frac{y}{t^2}\right)^{n/2} J_m (2t\sqrt{x}) \, J_n (2t\sqrt{y}) \, dt$$

$$= p^{-m} q^{-n} \, \Gamma\left(\frac{\lambda+1}{2}\right) 2^{\lambda-1/2} \left(1+\frac{2}{p}+\frac{2}{q}\right)^{-(\lambda-1)/2},$$

$$R(\lambda) > -1, \; R(m, n) > -1. \tag{b}$$

Putting $\lambda = m+n+\lambda-1$ and t for t^2 we have

$$x^{m/2} \cdot y^{n/2} \int_0^\infty t^{\lambda/2-1} \cdot e^{-t/2} \cdot J_m (2\sqrt{xt}) \, J_n (2\sqrt{yt}) \, dt$$

$$= \Gamma\left(\frac{m+n+\lambda}{2}\right) p^{-m} q^{-n} \left(\frac{1}{2}+\frac{1}{p}+\frac{1}{q}\right)^{-(\lambda+m+n)/2},$$

$$R(\lambda+m+n) > 0, \; R(m, n) > -1.$$

or $$\frac{x^m y^n}{\Gamma(m+1)\,\Gamma(n+1)} \psi_2 \left(\frac{m+n-\lambda}{2}; m+1, n+1; 2x, 2y\right)$$

$$= p^{-m} q^{-n} \left(1+\frac{2}{p}+\frac{2}{q}\right)^{-(\lambda+m+n)/2} \tag{3}$$

More generally we have

$$\frac{x_1^{m_1} \ldots x_k^{m_k}}{\Gamma(m_1+1) \ldots \Gamma(m_k+1)} \psi_2 \left(\frac{\lambda+m_1+\ldots+m_k}{2}; m_1+1, \ldots, m_k+1; 2x_1, \ldots, 2x_k\right)$$

$$= p_1^{-m_1} \ldots p_k^{-m_k} \left(1+\frac{2}{p_1}+\ldots+\frac{2}{p_k}\right)^{-(\lambda+m_1+\ldots+m_k)/2} \tag{4}$$

Again from (b), after putting $n=m$ and $\lambda=2m+1$, we have

$$\int_0^\infty t.\ e^{-t^2/2}\ (xy)^{m/2}\ J_m\ (2t\sqrt{x})\ J_m\ (2t\sqrt{y})\ dt$$

$$\doteq 2^m\ (pq)^{-m}\ \Gamma\ (m+1)\left(1+\frac{2}{p}+\frac{2}{q}\right)^{-m-1},$$

$$R\ (m) > -1.$$

$$\frac{2^{-m}\ (xy)^{m/2}\ e^{-2\,(x+y)}\ I_m\ (4\sqrt{xy})}{\Gamma\ (m+1)} \doteq (pq)^{-m}\ \left(1+\frac{2}{p}+\frac{2}{q}\right)^{-m-1} \tag{5}$$

When $m=0$, we get the result given in Operational Calculus in Two Variables by Ditkin and Prudnikov p. 56. A slight mistake has been corrected. See also p. 100. Vide also p. 208 [5].

(2) Consider the integral

$$\int_0^\infty x^\lambda\ e^{x/2}\ W_{k,u}\ (x)\ {}_mF_n\left(\begin{matrix}a_1,\ \ldots,\ a_m\ ;\\ b_1,\ \ldots,\ b_n\ ;\end{matrix}\ -zx\right) dx$$

$$= \frac{\prod_{j=1}^{n}\ \Gamma\ (b_j)}{\prod_{j=1}^{m}\ \Gamma\ (a_j)\ \Gamma\ (\tfrac{1}{2}-u-k)\ \Gamma\ (\tfrac{1}{2}+u-k)} \times$$

$$\times\ G^{m+2,\ 2}_{n-2,\ m+2}\left(\frac{1}{z}\left|\begin{matrix}1,\ 2+\lambda+k,\ b_1,\ \ldots.\ b_n\\ a_1,\ \ldots,\ a_m,\ u+\lambda+3/2,\ \lambda-u+3/2\end{matrix}\right.\right)$$

when $m=n,\ n+1$: $R\ (\lambda\pm u) > -3/2$, $R\ (-\lambda-k+a_j) > 1$

when $m=n-1$: $R\ (\lambda-k-\tfrac{1}{2}\ \Sigma a_j+\tfrac{1}{2}\ \Sigma b_j) > 5/4$, $R\ (\lambda\pm u) > -3/2$.

On writing $-\mathrm{N}$ for k, $1/2-u$ for u and putting $z=(pt)^{-1}$, $x=p_1 t$ and multiplying both the sides by $p^{1-\mathrm{N}}\, q^{u}\, \Gamma(2\mathrm{N})\, \Gamma(\mathrm{N}-u+1)$, we get after interpretation

$$x^{\mathrm{N}-1}\; y^{-\mu} \int_0^{\infty} t^{\lambda}\; e^{-xy/qt}\, \mathrm{M}_{u,\,\mathrm{N}-\frac{1}{2}}\left(\frac{xy}{t}\right) {}_m\mathrm{F}_n \left(\begin{matrix} a_1, \ldots, a_m ; \\ b_1, \ldots, b_n ; \end{matrix}\; t \right) dt$$

$$= \frac{\prod_{j=1}^{n} \Gamma(b_j)\, \Gamma(2\mathrm{N})}{\prod_{j=1}^{m} \Gamma(a_j)\, \Gamma(\mathrm{N}+\mu)}\; p^{\lambda-\mathrm{N}}\; q^{\mu-\lambda-1}$$

$$\mathrm{G}^{m+2,\,2}_{n+2,\,m+2}\left(pq \,\middle|\, \begin{matrix} 1,\, 2+\lambda-\mathrm{N},\, b_1, \ldots, b_n \\ a_1, \ldots, a_m,\, 2-\mu+\lambda,\, 1+\mu+\lambda \end{matrix} \right),$$

$m=n,\, n+1 : \mathrm{R}\,(-\lambda+\mathrm{N}+a_j) > 1$, $\mathrm{R}\,(\lambda+\mu) > -1$, $\mathrm{R}\,(\mathrm{N}) > 0$, $\mathrm{R}\,(\mathrm{N}-\mu) > -1$;

$m=n-1 : \mathrm{R}\,(\mathrm{N}-\lambda+\frac{1}{2}\Sigma a_j - \frac{1}{2}\Sigma b_j) > 3/4$, $\mathrm{R}\,(\mathrm{N}) > 0$, $\mathrm{R}\,(\mathrm{N}-\mu) > -1$;

or $\quad x^{\lambda+\mathrm{N}}\; y^{\lambda-\mu+1}\; \mathrm{G}^{2,\,m+1}_{m+1,\,n+3}$

$$\left(xy \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_m,\, -\mu-\lambda \\ 0,\, \mathrm{N}-\lambda-1,\, -\mathrm{N}-\lambda,\, 1-b_1, \ldots, 1-b_n \end{matrix} \right) = p^{\lambda-\mathrm{N}}\; q^{\mu-\lambda-1}\, .$$

$$\mathrm{G}^{m+2,\,2}_{n+2,\,m+2}\left(pq \,\middle|\, \begin{matrix} 1,\, 2+\lambda-\mathrm{N},\, b_1, \ldots, b_n \\ a_1, \ldots, a_m,\, 2-\mu+\lambda,\, 1+\mu+\lambda \end{matrix} \right) \qquad (6)$$

From (6) we can deduce

$$2\pi^{\frac{1}{2}}\, x^{\mathrm{N}-\frac{1}{2}}\, y^{\frac{1}{2}}\, \mathrm{I}_{\mathrm{N}-\frac{1}{2}}(\sqrt{xy})\; \mathrm{K}_{\mathrm{N}-\frac{1}{2}}(\sqrt{xy}) = p^{\frac{1}{2}-\mathrm{N}}\; q^{-\frac{1}{2}}\; \mathrm{G}^{2,2}_{2,2}$$

$$\left(pq \,\middle|\, \begin{matrix} 1,\, 3/2-\mathrm{N} \\ 3/2,\, \frac{1}{2} \end{matrix} \right),\; \mathrm{R}\,(\mathrm{N}) > 0 \qquad (7)$$

$$p^{-\lambda+\frac{1}{4}}\, q^{\mu-\frac{1}{4}}\, e^{\sqrt{pq}}\, W_{-2\lambda-\frac{1}{2},\, 2\mu-1}\, (2\sqrt{pq})$$

$$\doteq \frac{2^{2\lambda-\frac{1}{2}}\, \pi^{-3/2}\, x^{2\lambda+\frac{1}{2}}\, y^{\lambda-\mu+1}}{\Gamma(2+2\lambda-\mu)\,\Gamma(2\lambda+2\mu)}\, G^{2,\,3}_{3,\,3}\left(xy\,\middle|\,\begin{matrix}\mu-\lambda-\frac{1}{2},\ \frac{1}{2}-\lambda-\mu,\ -\mu-\lambda\\ 0,\ -\frac{1}{2},\ -2\lambda-\frac{1}{2}\end{matrix}\right)$$

$$R(\lambda) > -\tfrac{1}{2},\ R(\lambda-\mu) > -3/2. \qquad (8)$$

Let $\mu=\lambda+\frac{1}{2}$

$$p^{-\lambda+\frac{1}{4}}\, q^{\lambda+\frac{1}{4}}\, e^{\sqrt{pq}}\, W_{-2\lambda-\frac{1}{2},\, 2\lambda}\, (2\sqrt{pq})$$

$$\doteq \frac{2^{2\lambda-\frac{1}{2}}\, x^{2\lambda+\frac{1}{2}}\, y^{\frac{1}{2}}\, \pi^{-3/2}}{\Gamma(4\lambda+1)}\, G^{2,\,2}_{2,\,2}\left(xy\,\middle|\,\begin{matrix}0,\ -2\lambda\\ 0,\ -\frac{1}{2}\end{matrix}\right),\ R(\lambda) > -\tfrac{1}{2}. \qquad (9)$$

(3) In the integral

$$\int_0^\infty \frac{t^{-\lambda-k-2}}{(p+t)^{\mu-k+1/2}}\, {}_1F_1\left(a\,;\,\mu+\lambda+3/2\,;\,-\frac{1}{t}\right) dt$$

$$= \frac{\Gamma(\lambda+\mu+3/2)\,\Gamma(a-\lambda-k-1)\, e^{1/2p}}{\Gamma(\mu-k+\frac{1}{2})\, p^{\lambda+2\mu-k+1/2}}\, W_{\lambda+k+2-2a/2,\ \lambda+k+1/2}\left(\frac{1}{p}\right).$$

$$R(\lambda+\mu) > -3/2,\ R(a-\lambda-k) > 1.$$

putting pq for p, $k=\mu-\frac{1}{2}$, $a=\lambda+\mu+3/2=1$, we get

$$\int_0^\infty \frac{e^{-1/t}}{t}\cdot\frac{pq}{(pq+t)}\, dt = (pq)^{\frac{1}{2}}\, e^{1/2pq}\, W_{-\frac{1}{2},\, 0}\left(\frac{1}{pq}\right)$$

Now after interpretation we get

$$\int_0^\infty e^{-1/t} \cdot t^{-1} \, J_0 (2\sqrt{xyt}) \, dt \doteq (pq)^{-\frac{1}{2}} \, e^{1/2pq} \; W_{\frac{1}{2}, 0} \left(\frac{1}{pq}\right)$$

or $$G^{2,\,0}_{0,\,3}\left(xy\Big|_{0,\,0,\,0}\right) \doteq (pq)^{-\frac{1}{2}} \, e^{1/2pq} \; W_{-\frac{1}{2},0}\left(\frac{1}{pq}\right). \tag{10}$$

Similarly from (c) we have

$$\int_0^\infty \frac{q\, t^{-\lambda-k-2} \; e^{-1/t}}{(pq+t)} dt = \Gamma(\lambda+k+2)\, q\, (pq)^{\lambda+k+2/2}$$

$$e^{1/2xq} \; W_{-\lambda+k+2/2,\ \lambda+k+1/2} \left(\frac{1}{pq}\right);$$

which gives on interpretation after putting $\lambda+k+2=m>0$,

$$\frac{1}{y\,\Gamma(m)} \, G^{2,\,0}_{0,\,3}\left(xy\Big|_{1,\,m,\,0}\right) \doteq q\;(pq)^{-m}$$

$$e^{1/2pq} \; W_{-m/2,\ m-1/2}\left(\frac{1}{pq}\right); \; m > 0. \tag{11}$$

(4) Now let us consider the integrals

$$\int_0^\infty x^{\lambda} \quad e^{-x^2/4} \, D_{-\nu}(x) \; {}_mF_n \left(\begin{matrix} a_1, \ldots, a_m\,; \\ b_1, \ldots, b_n\,; \end{matrix} -z^2x^2\right) dx$$

$$= \frac{2^{\lambda-v-1/2}\,\Gamma\left(\frac{\lambda+2}{2}\right)\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda+v+2}{2}\right)}$$

$${}_{m+2}F_{n+1}\left|\begin{matrix} a_1, \ldots, a_m, \frac{\lambda+1}{2}, \frac{\lambda+2}{2}; \\ b_1, \ldots, b_n, \frac{\lambda+v+2}{2}; \end{matrix} \quad -2z^2\right|, \ R(\lambda) > -1; \qquad (d)$$

$$\int_0^\infty x^\lambda\, e^{-x/2-zx}\, M_{k,\mu}(x)\, dx$$

$$= \frac{\Gamma(2\mu+1)\,\Gamma(\mu+\lambda+3/2)\,\Gamma(k-\lambda-1)}{\Gamma(\mu+k+1/2)\,\Gamma(\mu-\lambda-1/2)}\, {}_2F_1\left(\begin{matrix}\mu+\lambda+3/2, 3/2, +\lambda-\mu; \\ 2+\lambda-k;\end{matrix} \ -z\right) +$$

$$\frac{\Gamma(1+\lambda-k)\,\Gamma(2\mu+1)}{\Gamma(\mu-k+1/2)}\, z^{k-\lambda-1}\, {}_2F_1\left(\begin{matrix}\frac{1}{2}+k-\mu, \mu+k+\frac{1}{2}; \\ k-\lambda;\end{matrix} \ -z\right),$$

$$R(\lambda+\mu) > -3/2. \ (e)$$

In (d) putting $x = t/z$, $v = -2M$ and then $z = pq^{1/2}$ and multiplying both the sides by $(-)^M\,\Gamma(1/2)\,p^{-4M}\,q^{\frac{1}{2}-M}$, we get after interpretation

$$x^{2M}\, y^{-1/2} \int_0^\infty t^{\lambda-2M}\, bc_{4M}\left(2^{5/4}\, y^{1/4}\, x^{1/2}\, t^{1/2}\right)\, {}_mF_n\left(\begin{matrix}a_1, \ldots, a_m; \\ b_1, \ldots, b_n;\end{matrix} \ -t^2\right) dt$$

$$\doteq \frac{\pi^{1/2}\, 2^{\lambda+2M-1/2}\,\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(\frac{\lambda+2}{2}\right)}{\Gamma\left(\frac{\lambda-2M+2}{2}\right)}\, (-1)^M\, p^{1+\lambda-4M}\, q^{1+\lambda/2-M}$$

$${}_{m+2}F_{n+1}\left(\begin{matrix}a_1, \ldots, a_m, \lambda+1/2, \lambda+2/2; \\ b_1, \ldots, b_n, \lambda-2M+2/2;\end{matrix} \ -2p^2q\right)$$

From this, after putting $M=0$, $m=n$, $a's=b's$, we have

$$-1/2 \int_0^\infty t^{\lambda} e^{-t^2} \text{ ber } \left(2^{5/4} y^{1/4} x^{1/2} t^{1/2} \right) dt$$

$$\therefore = \pi^{\frac{1}{2}} 2^{(\lambda-1)/2} \Gamma \left(\frac{\lambda+1}{2}\right) p^{1+\lambda} q^{1+\lambda/2} (1+2p^2q)^{-(\lambda+1)/2}, \ R(\lambda) > -1.$$

or $$\frac{y^{-1/2} 2^{-\lambda+1/2} \pi^{-1/2}}{\Gamma(\lambda+1)}$$

$$\sum_{r=0}^{\infty} \frac{(-1)^r (2x^2y)^r \Gamma\left(\frac{\lambda+1}{2}+r\right)}{[\Gamma(2r+1)]^2} \therefore = \frac{p^{\lambda+1} q^{\lambda+\frac{1}{2}}}{(1+2p^2q)^{(\lambda+1)/2}}. \tag{12}$$

Similarly we can prove

$$\frac{\pi^{-1/2} 2^{-\lambda/2-1/2}}{\Gamma\left(\frac{\lambda+2}{2}\right)}$$

$$\sum_{r=0}^{\infty} \frac{(-)^r (2x^2y)^{r+\frac{1}{2}} \Gamma\left(\frac{\lambda+2}{2}+r\right)}{[\Gamma(2r+2)]^2} \therefore = \frac{p^{\lambda+1} q^{(\lambda+1)/2}}{(1+2p^2q)^{(\lambda-2)/2}}, \ R(\lambda) > -2. \tag{13}$$

Again in (e) patting $z=pq$, $x=t/(pq)$ aud multiplying both the sides by $(pq)^{1-k}$, after interpretation, we get

$$(xy)^{2k-1/3} \int_0^\infty t^{3\lambda-k+2/3} e^{-t} J_{k+\mu-\frac{1}{2},\, 2\mu}\left(3\sqrt[3]{xy\,t}\right) dt =$$

$$\frac{\Gamma(\lambda+\mu+3/2)\,\Gamma(k-\lambda-1)}{\Gamma(\mu-\lambda-\frac{1}{2})} (pq)^{2-k+\lambda}\ {}_2F_1\left(\begin{matrix}\mu+\lambda+3/2,\ 3/2+\lambda-\mu\ ; \\ 2+\lambda-k\ ;\end{matrix} -pq\right) +$$

$$\frac{\Gamma(1+\lambda-k)\,\Gamma(\mu+k+\frac{1}{2})}{\Gamma(\mu-k+\frac{1}{2})}\ pq\ {}_2F_1\left(\begin{matrix}\frac{1}{2}+k-\mu,\ \mu+k+\frac{1}{2}; \\ k-\lambda\ ;\end{matrix} -pq\right),\ R(\lambda+\mu) > -3/2,$$

$$R(k+\mu) > -\tfrac{1}{2}.$$

or
$$(xy)^{k+\mu-\frac{1}{2}}\ {}_1F_2\left(\begin{matrix}\lambda+\mu+3/2 \\ 2\mu+1,\ k+\mu+\frac{1}{2}\ ;\end{matrix} -xy\right)$$

$$= \frac{\Gamma(k+\mu+\frac{1}{2})\,\Gamma(2\mu+1)\,\Gamma(k-\lambda-1)}{\Gamma(\mu-\lambda-\frac{1}{2})} (pq)^{\lambda-k+2} \times$$

$${}_2F_1\left(\begin{matrix}\mu+\lambda+3/2,\ 3/2+\lambda-\mu\ ; \\ 2+\lambda-k\ ;\end{matrix} -pq\right) + \frac{\Gamma(2\mu+1)\,\Gamma(1+\lambda-k)\,[\Gamma(k+\mu+1)]^2}{\Gamma(\lambda+\mu+3/2)\,\Gamma(\mu-k+\frac{1}{2})} pq$$

$${}_2F_1\left(\begin{matrix}\frac{1}{2}+k-\mu,\ \mu+k+\frac{1}{2}\ ; \\ k-\lambda\ ;\end{matrix} -pq\right). \qquad (14)$$

From this we can deduce following interesting particular cases***.

$$\pi^{\frac{1}{2}}\ (xy)^{a+m-\frac{1}{2}} \sum_{r=0}^{m} \binom{m}{r} \frac{\Gamma(2a-1+r)\,\Gamma(a+\frac{1}{2}+r)}{\Gamma(a-\frac{1}{2}+r)\,\Gamma(2a+m+r)}\ J^2_{a-\frac{1}{2}+r}\ (\sqrt{xy})$$

$$= \frac{\Gamma(a+m)\,\Gamma(a)}{2\,\pi^{1/2}} (pq)^{1-a-m}\ {}_2F_1\left(\begin{matrix}a,\ \frac{1}{2}\ ; \\ 1-a-m\ ;\end{matrix} -pq\right)$$

$$+ \frac{\Gamma(-a-m)\,\Gamma(2a+m)}{2\,\Gamma(\frac{1}{2}-a-m)}\ pq\ {}_2F_1\left(\begin{matrix}a+m+\frac{1}{2},\ 2+m \\ a+m+1\ ;\end{matrix} -pq\right),$$

$R(a) > 0$, $R(2a+m) > 0$ and $a+m$ is not an integer. (15)

*** A similar result is given in [4], p. 141, but factor x on the left in (16) is absent, The results (15), (16) and (17) are believed to be new.

$$(xy)^a\ J^2(\sqrt{xy}) \doteq \frac{2^{-2a}\,\Gamma(2a+1)\,\Gamma(a+\frac{1}{2})}{\Gamma(a+1)\,\Gamma(\frac{1}{2})}\,(pq)^{\frac{1}{2}-a}\,{}_2F_1\left(\begin{matrix}a+\frac{1}{2},\ \frac{1}{2}\ ;\\ \frac{1}{2}-a\ ;\end{matrix}\ -p\right)$$

$$+\ \frac{\Gamma(2a+1)\,\Gamma(-a-\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(-a)}\ \times$$

$$pq\ {}_2F_1\left(\begin{matrix}a+1,\ 2a+1\ ;\\ a+3/2\ ;\end{matrix}\ -pq\right),\ R\ (a) > -\tfrac{1}{2} \text{ and } a+\tfrac{1}{2} \text{ is not an integer.} \tag{16}$$

$$(xy)^{a/2}\quad H_a\,(2\sqrt{xy}) \doteq \frac{\Gamma(a+\frac{1}{2})}{\Gamma(\frac{1}{2})}\,(pq)^{\frac{1}{2}-a}\,{}_2F_1\left(\begin{matrix}1,\ \frac{1}{2}\ ;\\ \frac{1}{2}-a\ ;\end{matrix}\ -pq\right)$$

$$+\ \frac{\Gamma(-a-\frac{1}{2})\,\Gamma(a+3/2)\,pq}{\Gamma(-a)\,(1+pq)^{a+1}},$$

$$R\ (a) > -\ 3/2 \text{ and } a+\tfrac{1}{2} \text{ is not an integer.} \tag{17}$$

5. Now we shall evaluate one finite integral containing $W_{k,m}(x)$ with the help of Laplace transformation in two variables.

We know

$$\frac{q}{q+p-b} \doteq \begin{cases} o \text{ for } y > x \\ e^{by} \text{ for } y < x \end{cases} \tag{f}$$

From (c) we have

$$\int_0^\infty \frac{t^{-m}\ e^{1/t}}{p\,(p+q+t)}\,dt = \frac{\Gamma(m)}{p\,(p+q)^{m/2}}\ e^{\frac{1}{2(p+q)}}\ W_{-m/2,m-1/2}\left(\frac{1}{p+q}\right)$$

Now with the help of (f) we get

$$\text{L. H. S.} = \int_0^\infty t^{-m}\ e^{-1/t}\,dt \int_0^\infty \int_0^\infty e^{-px-qy}\,\phi\,(x, y, t)\,dx\,dy,$$

where $\phi(x, y, t) = 0$ for $y >$

$= e^{-ty}$ for $y < x$

$$= \int_0^\infty t^{-m} e^{-1/t}\, dt \left[\int_0^\infty e^{-px}\, dx \left\{ \int_0^x e^{-qy}\, \phi(x, y, t)\, dy + \int_0^\infty e^{-qy}\, \phi(x, y, t)\, dy \right\} \right]$$

$$= \int_0^\infty t^{-m} e^{-1/t}\, dt \left[\int_0^\infty e^{-px}\, dx \int_0^x e^{-qy-ty}\, dy \right]$$

$$= \int_0^\infty t^{-m} e^{-1/t}\, dt \left[\int_0^\infty e^{-px}\, x.\, dx \int_0^1 e^{-(q+t)\, xy}\, dy \right]$$

$$= \int_0^\infty t^{-m} t^{-1/t}\, dt \left[\int_0^1 dy \int_0^\infty x.\, e^{-x\{p+(q+t)\, y\}}\, dx \right]$$

$$= \int_0^\infty t^{-m} e^{-1/t}\, dt \int_0^1 \frac{dy}{\{p+(q+t)y\}^2} = \int_0^1 y^{-2}\, dy \int_0^\infty \frac{t^{-m}.\, e^{-1/t}}{\left(\frac{p+qy}{y}+t\right)^2}\, dt.$$

$$= \Gamma(m+1) \int_0^1 e^{y/2(p+qy)}\, y^{m-2/2}\, (p+qy)^{-m+2/2}$$

$$W_{-m+2/2,\, m-1/2} \left(\frac{y}{p+qy}\right) dy$$

$$\text{or} \int_0^1 y^{m-2/2} \; (p+qy)^{-m+2/2} \; e^{y/2(p+qy)} \, W_{-m+2/2,m-1/2}\left(\frac{y}{p+qy}\right) dy$$

$$= \frac{exp\left(\frac{1}{2p+2q}\right)}{mp\,(p+q)^{m/2}} W_{-m/2,m-2/2}\left(\frac{1}{p+q}\right)$$

Putting $q = p$ we get

$$\int_0^1 y^{m-2/2} \,(1+y)^{-m+2/2} \; e^{y/2p\,(1+y)} W_{-m+2/2,m-1/2}\left(\frac{y}{p\,(1+y)}\right) dy$$

$$= \frac{e^{1/(4p)}}{m\, 2^{m/2}} W_{-m/2,m-1/2}\left(\frac{1}{2p}\right), \; R\,(m) > o.$$

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INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

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ABSTRACT

In this paper a few infinite integrals involving hypergeometric functions of two variables have been evaluated in terms of Lauricellas' function F_c and Appells' function F_4.

1. *Introduction* :—The object of this paper is to evaluate a few integrals involving hypergeometric functions of two variables by applying the parseval—Goldstein (3) theorem of operational calculus ; that if

$$\phi(p) \doteqdot h(t)$$

and

$$\psi(p) \doteqdot g(t)$$

then

$$(1.1) \quad \int_0^\infty t^{-1} \phi(t)\, g(t)\, dt = \int_0^\infty t^{-1} h(t)\, \psi(t)\, dt,$$

when the integrals are convergent.

The notation $\phi(p) \doteqdot h(t)$ stands for the Laplaces' integral ; that is

$$\phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt,$$

where $R(p) > 0$ and the integral is convergent.

We require the following results (1, p. 187, eqn. 43, 217, 198).

$$(1.2) \quad t^{v-1}\, I_{2\mu}(2\sqrt{ct})\, I_{2\rho}(2\sqrt{ct} \doteqdot \frac{c^{\mu}\; c^{\rho}\; \Gamma(\mu+v+\rho)}{\Gamma(1+2\mu)\;\Gamma(1+2\rho)} \times$$

$$p^{-v-\mu-\rho+1}\psi_2\left(v+\rho+\mu\,;\,1+2\mu,\,1+2\rho\,;\,\frac{c}{p},\,\frac{c}{p}\right)$$

valid for $R(p) > 0$, $R(v+\rho+\mu) > 0$,

(1.2) has been derived from (1, p. 187, eqn. 43).

$$(1.3) \quad t^{v-1}\, J_{2\mu}(2\sqrt{ct})\, J_{2\rho}(2\sqrt{ct}) \doteqdot \frac{c^{\mu}\; c^{\rho}\; \Gamma(\mu+v+\rho)}{\Gamma(1+2\mu)\;\Gamma(1+2\rho)}$$

$$p^{-v-\mu-\rho+1}\quad \psi_2\left(v+\rho+\mu\,;\,1+2\mu,\,1+2\rho\,;\,-\frac{c}{p},\,-\frac{c}{p}\right),$$

valid for $R(p) > 0$, $R(v+\rho+\mu) > 0$.

(1.3) has been derived from (1, p. 187, eqn. 43),

$$(1.4) \quad t^{-k}\quad e^{-a/2t}\, W_{k,\,m}(a/t) \doteqdot (2\sqrt{a}\; p^{k+\frac{1}{2}}\; k_{2m}\quad (2\sqrt{ap}),$$

valid for $R(a) > 0$.

$$(1.5) \quad t^{-m-\frac{1}{2}}\; k_n(t) \doteqdot \sqrt{\pi/2}\;\; \Gamma(\tfrac{1}{2} - m \pm n)\, p\,(p-1)^{m/2}\;\; P^{m}_{n-\frac{1}{2}}\;(p)$$

valid for $R(\frac{1}{2} - m \pm n) > 0$, $R(p+1) > 0$.
Also (4, p. 342)

$$(1.6) \quad t^{m-\frac{1}{2}}\; I_n(t) \doteqdot \sqrt{2/\pi}\;\; p\,(p^2-1)^{-m/2}\; Q^{m}_{n-\frac{1}{2}}\quad (p),$$

valid for $R(\frac{1}{2} + m + n) > 0$, $R(p) > 1$.

and (5, p. 222)

$$(1.7) \quad t^{-\lambda-1}\,\psi_2(\lambda+1\,;\,\lambda+1,\,\delta+1\,;\,-a/t,\,-b/t) \doteqdot$$

$$\frac{2\,\Gamma\;(\delta+1)}{a^{\lambda/2}\quad b^{\delta/2}}\; p^{\lambda/2\,-\,\delta/2\,+1}\; K_{\lambda}\;(2\sqrt{ap})\;\; I_{\delta}\;(2\sqrt{bp})$$

valid for R $(p) > 0$, R $(a) > 0$, R $(b) > 0$,

$$(1.8)\quad t^{\lambda}\, F_3\left(\tfrac{1}{2}+\rho, \tfrac{1}{2}+\mu, \tfrac{1}{2}-\rho, \tfrac{1}{2}-\mu ; 1+\lambda ; -t/2a ; -t/2b\right)$$

$$\doteqdot \frac{2\sqrt{ab}\; p^{1-\lambda}\,\Gamma(1+\lambda)}{\pi}\, e^{p(a+b)}\, k_{\rho}(ap)\, k_{\mu}(bp)$$

valid for R $(1+\lambda) > 0$, R $(a) > 0$, R $(b) > 0$,
in the investigation.

INTEGRALS

2. The first of the integrals to be proved is

$$(2.1)\quad \int_0^{\infty} t^{\lambda+v+\rho+\mu-1}\, \psi_2(\lambda+1 ; \lambda+1, \delta+1 ; -at, -bt)$$

$$\psi_2(v+\mu+\rho ; 1+2\mu, 1+2\rho ; ct, ct)\, dt = \frac{\Gamma(\lambda+v+\mu+\rho)}{a^{\lambda+v+\mu+\rho}} \times$$

$$F_c\,[\lambda+v+\mu+\rho, v+\mu+\rho ; 1+\delta, 1+2\mu, 1+2\rho ; b/a, c/a, c/a],$$

valid for R $(\lambda+v+\mu+\rho) > 0$, R $(\sqrt{a} \pm \sqrt{b})^2 >$ R $(\sqrt{c} \pm \sqrt{e})^2$.
To proved (2.1), we apply (1.1) to the operational pairs (1.2) and (1.7), we get

$$\int_0^{\infty} t^{\lambda+\mu+v+\rho-1}\, \psi_2(\lambda+1 ; \lambda+1, \delta+1 ; -at, -bt) \times$$

$$\psi_2(v+\mu+\rho ; 1+2\mu, 1+2\rho ; ct, et)\, dt = \frac{4\,\Gamma(1+2\rho)\,\Gamma(1+2\mu)\,\Gamma(1+\delta)}{\Gamma(v+\rho+\mu)\, c^{\mu}\, e^{\rho}\, a^{\lambda/2}\, b^{\delta/2}}$$

$$\int_0^{\infty} t^{\lambda-\delta+2v-1}\, K_{\lambda}(2\sqrt{at})\, I_{\delta}(2\sqrt{bt})\, I_{2\mu}(1\sqrt{ct})\, I_{2\rho}(2\sqrt{et})\, dt.$$

The integral on the right can now be evaluated on using Sharmas' result (6) to get (2.1).

We mention below some of the particular cases of the general result (2.1).

(i) If $b=c$, then (2.1) reduces to a known result (1, p. 223).

(ii) If $e=o$ and $v=o$, then (2.1) yields a new integral for Appells' function F_4

$$(2.2) \quad \int_0^\infty t^{\lambda+\mu+\rho-1}\, \psi_2(\lambda+1\,;\,\lambda+1,\,\delta+1\,;\,-at,\,-bt)$$

$$ {}_1F_1(\mu+\rho\,;\,1+2\mu\,;\,ct)\,dt = \frac{\Gamma(\lambda+\rho+\mu)}{a^{\lambda+\mu+\rho}} \times$$

$$F_4[\lambda+\mu+\rho,\,\mu+\rho\,;\,1+\delta,\,1+2\mu\,;\,b/a,\,c/a],$$

valid for $R(\lambda+\mu+\rho)>0$, $R(\sqrt{a}\pm\sqrt{b})^2 > R(c)$,
using the relation (2, p. 430)

$$ {}_1F_1(\tfrac{1}{2}-k+m\,;\,1+2m\,;\,x) = x^{-m-\frac{1}{2}}\, e^{\frac{1}{2}x}\, M_{k,\,m}(x)$$

and the well known relation

$$(2.3) \quad \psi_2(v+1\,;\,v+1,\,v+1\,;\,\pm x,\,\pm y) = \Gamma(v+1)\,(xy)^{-\frac{1}{2}v} e^{\pm(x+\lambda)}\, I_v(2\sqrt{xy}),$$

in (2.2), we get a known result (7).

(iii) Assuming $v=1$, $\mu=\delta$, $\lambda=\rho$, and using the relation (2.3) in (2.1) it reduces to

$$\int_0^\infty e^{-(a+b-c-e)t}\, I_\lambda(2\sqrt{ab}\,t)\, I_\mu(2\sqrt{ce}\,t)\,dt$$

$$= \frac{\Gamma(\lambda+\mu+1)\,(ab)^{\frac{1}{2}\lambda}\,(ce)^{\frac{1}{2}\mu}}{\Gamma(\lambda+1)\;\Gamma(\mu+1)\;a^{\lambda+\mu+1}} \times$$

$$F_c[1+\lambda+\mu\,;\,1+\mu\,;\,1+\lambda,\,1+\mu,\,1+\mu\,;\,b/a,\,c/a,\,e/a].$$

On evaluating the integral with the help of the result (1, p. 196).
We obtain an interesting relation between F_c and F_4.

$$(2.4) \quad F_c[1+2\lambda+2\mu,\,1+2\mu\,;\,1+2\lambda,\,1+2\mu,\,1+2\mu,\,x,\,y,\,z]$$

$$= (1+x-y-z)^{-1-2\lambda-2\mu}\, F_4\Big[\tfrac{1}{2}+\lambda+\mu,\,1+\lambda+\mu\,;\,1+2\lambda,\,1+2\mu\,;\,\frac{4x}{(1+x-y-z)^2}\,\frac{4yz}{(1+x-y-z)^2}\Big].$$

The following integrals can be evaluated in the same way from the pair of the formulae (1.3) (1.4), (1.5) (1.8) and (1.6) (1.8).

$$(2.5)\quad \int_0^\infty t^{k+v+\mu+\rho-2}\,\psi_2\,(v+\mu+\rho\,;\,1+2\mu,\ 1+2\rho\,;\,-ct,\ -et)$$

$$e^{-\frac{1}{2}at}\,W_{k,\,m}\,(at)\,dt = \frac{\Gamma\,(k+v+\mu+\rho\pm m-\frac{1}{2})}{\Gamma\,(v+\mu+\rho)}\,a^{1-k-v-\mu-\rho}$$

$$F_4\,[k+v+\mu+\rho-m-\tfrac{1}{2},\ k+v+\mu+\rho+m-\tfrac{1}{2}\,;\,1+2\mu,\ 1+2\rho\,;\,-c/a,\ -e/a],$$

valid for $R\,(k+v+\mu+\rho\pm m-\frac{1}{2}) > 0$, $R\,[a+(\sqrt{c}\pm\sqrt{e})^2] > 0$.

Assuming $v=1$, $\mu=\rho$ and using the formula (2.3) in (2.7), it reduces to a known result (7).

$$(2.6)\quad \int_0^\infty t^{\lambda}\,[\,(t+a+b)^2-1\,]^{m/2}\,P^{m}_{n-\frac{1}{2}}\,(t+a+b)$$

$$F_3\left[\tfrac{1}{2}+\rho,\ \tfrac{1}{2}+\mu\,;\,\tfrac{1}{2}-\rho,\ \tfrac{1}{2}-\mu\,;\,1+\lambda\,;\,-\frac{t}{2b},\ -\frac{t}{2b}\right]dt$$

$$=\sum_{\rho,\,-\rho}\ \sum_{\mu,\,-\mu}\frac{\Gamma\,(-\rho)\,\Gamma\,(-\mu)\,\Gamma\,(1+\lambda)\,\Gamma\,[\frac{1}{2}\,(\rho+\mu-\lambda-m\pm n+\frac{1}{2})\,]}{(\pi)^{3/2}\,2^{\lambda+m+2}\quad\Gamma\,(\frac{1}{2}-m\pm n)\,a^{-\rho-\frac{1}{2}}\,b^{-\mu-\frac{1}{2}}}\times$$

$$F_4\,[\tfrac{1}{2}\,(-m-n+\mu+\rho-\lambda+\tfrac{1}{2}),\ \tfrac{1}{2}\,(-m+n+\mu+\rho-\lambda+\tfrac{1}{2})\,;\,1+\rho,\ 1+\mu\,;\,-a^2,\ -b^2],$$

valid for $R\,(1+\lambda)>0$, $R\,(-m\pm n\pm\mu\pm\rho-\lambda) > -\frac{1}{2}$, $R\,(a+b)>1$.

$$(2.7)\quad \int_0^\infty t^{\lambda}\,[\,(t+a+b)^2-1]^{-m/2}\,Q^{m}_{n-1/2}\,(t+a+b)$$

$$F_3\left[\tfrac{1}{2}+\rho,\ \tfrac{1}{2}+\mu,\ \tfrac{1}{2}-\rho,\ \tfrac{1}{2}-\mu\,;\,1+\lambda\,;\,-\frac{t}{2a},\ -\frac{t}{2b}\right]dt$$

$$=\sum_{\mu,\,-\mu}\frac{b^{\mu+\frac{1}{2}}\,\Gamma\,(1+\lambda)\,\Gamma\,(-\mu)\,\Gamma\,[\frac{1}{2}\,(m+n+\mu\pm\rho-\lambda+\frac{1}{2})\,]}{2^{\lambda-m+2}\quad\sqrt{\pi}\,a^{m+n+\mu-\lambda}\quad\Gamma\,(n+1)}$$

$$F_4\left[\tfrac{1}{2}(m+n+\mu-\rho-\lambda+\tfrac{1}{2}),\ \tfrac{1}{2}(m+n+\mu+\rho-\lambda+\tfrac{1}{2})\ ;\ 1+\mu,\ 1+\rho\ ;\ \frac{1}{a^2},\ \frac{b^2}{a^2}\right].$$

Valid for $R(1+\lambda)>0$, $R(m+n\pm\mu\pm\rho-\lambda)>-\frac{1}{2}$, $R(a+b)>1$.

In (2.9), we have used MacRoberts' definition of $\overset{m}{\underset{n}{Q}}(x)$.

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ON SELF-RECIPROCAL FUNCTIONS

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ABSTRACT

In this paper, a new generalisation of the Hankel transform has been introduced as

$$g(x)=2\beta\gamma\int_0^\infty (xy)^{\gamma-\frac{1}{2}}\, G^{q,p}_{2p,2q}\left[\beta^2(xy)^{2\gamma}\left|\begin{matrix}a_1,\ldots,a_p, & -a_1,\ldots,-a_p\\ b_1,\ldots,b_q, & -b_1,\ldots,-b_q\end{matrix}\right.\right] f(y)\,dy$$

with the help of a symmetrical Fourier kernel given by R. Narain. Various other generalisations are obtained as particular cases of this Two theorems have been established and some new self-reciprocal functions have been obtained.

1. Recently R. Narain (8, p. 951) has given a symmetrical Fourier kernel in terms of G-function as

$$(1.1)\qquad 2\,\beta\gamma x^{\gamma-\frac{1}{2}}\, G^{q,p}_{2p,\,2q}\left[\beta^2 x^{2\gamma}\left|\begin{matrix}a_1,\ldots,a_p, & -a_1,\ldots,-a_p\\ b_1,\ldots,b_q, & -b_1,\ldots,-b_q\end{matrix}\right.\right],$$

where β and γ are real constants.

With $\gamma=1$ and $\beta=\frac{1}{2}$ this can be reduced to another Fourier kernel (10, p. 298)

$$(1.2)\qquad \sqrt{x}\; G^{q,p}_{2p,\,2q}\left[\frac{x^2}{4}\left|\begin{matrix}a_1,\ldots,a_p, & -a_1,\ldots,-a_p\\ b_1,\ldots,b_q, & -b_1,\ldots,-b_q\end{matrix}\right.\right].$$

With the help of (1.1) a new generalisation of Hankel-transform may be introduced in the form

$$(1.3)\quad g(x)=2\beta\gamma\int_0^\infty (xy)^{\gamma-\frac{1}{2}}\, G^{q,p}_{2p,\,2q}\left[\beta^2(xy)^{2\gamma}\left|\begin{matrix}a_1,\ldots,a_p, & -a_1,\ldots,-a_p\\ b_1,\ldots,b_q, & -b_1,\ldots,-b_q\end{matrix}\right.\right] f(y)\,dy.$$

By using a simple identity (1, p. 209), (1.3) may be written in a more compact form

$$(1.4) \qquad g(x) = 2\gamma \beta^{1/2\gamma} \int_0^\infty G^{q,\,p}_{2p,\,2q}$$

$$\left[\beta^2(xy)^{2\gamma} \left| \begin{matrix} \frac{2\gamma-1}{4\gamma}+a_1, \ldots, \frac{2\gamma-1}{4\gamma}+a_p, \frac{2\gamma-1}{4\gamma} - a_1, \ldots, \frac{2\gamma-1}{4\gamma} - a_p \\ \frac{2\gamma-1}{4\gamma}+b_1, \ldots, \frac{2\gamma-1}{4\gamma}+b_q, \frac{2\gamma-1}{4\gamma} - b_1, \ldots, \frac{2\gamma-1}{4\gamma} - b_q \end{matrix} \right. \right] f(y)\, dy.$$

Further it has been given in (8, pp. 957-58) that (1.1) yields various other generalisations of the well know Hankel transform.

(A) With $\gamma = 1$, $\beta = 2^{-n}$, where n is a positive integer, and setting the parameters suitably (1.1) reduces to

$$(1.5) \qquad 2^{1-n}\, x^{\frac{1}{2}}\, G^{n,0}_{0,\,2n} \left[\frac{x^2}{2^{2n}} \middle| \frac{\mu_1}{2}, \ldots, \frac{\mu_n}{2}, -\frac{\mu_1}{2}, \ldots, -\frac{\mu_n}{2} \right] = \omega_{\mu_1, \ldots, \mu_n}(x),$$

where $\omega_{\mu_1, \ldots, \mu_n}(x)$ is the kernel defined by Bhatnagar (3, p. 43).

(i) when $n=1$, this gives the Hankel's kernel $x^{\frac{1}{2}}\, J_\nu(x)$.

(ii) If $n=2$, (1.5) gives

$$(1.6) \qquad \tfrac{1}{2}\, x^{\frac{1}{2}}\, G^{2,\,0}_{0,\,4} \left[\frac{x^2}{16} \middle| \frac{\mu}{2}, \frac{\nu}{2}, -\frac{\mu}{2}, -\frac{\nu}{2} \right] = \omega_{\mu,\,\nu}(x),$$

where $\omega_{\mu,\nu}(x)$ is the kernel, given by Watson (14, p. 308).

(B) Putting $\beta = \left(\frac{1}{2k}\right)^k$, $\gamma = k$ and giving suitable values to the parameters in (1.1) we obtain

$$(1.7) \qquad (2k)^{\frac{1}{2}} \left(\frac{x}{2k}\right)^{k-\frac{1}{2}} G^{k,\,0}_{0,\,2k} \left[\left(\frac{x}{2k}\right)^{2k} \right.$$

$$\left| \frac{\nu}{2k}, \frac{\nu+1}{2k}, \ldots, \frac{\nu+k-1}{2k}, \frac{\nu-k+1}{2k}, \ldots, \frac{\nu-2}{2k}, \frac{\nu-1}{2k}, \frac{1-2k}{2k} \right] = x^{\frac{1}{2}}\, J_{\nu,k}(x),$$

where k is a positive integer and $x^{\frac{1}{2}} J_{\nu, k}(x)$ is Everitt's kernel (5, p. 275)

(C) Substituting $\beta = \frac{1}{2}$, $\sqrt{\ } = 1$, $q = 2$, $p = 1$ and choosing the parameters suitably in (1.1) we get

$$\text{(1.8)} \qquad \sqrt{2}\; G^{2,1}_{2,4}\left[\frac{x^2}{4} \middle| \begin{matrix} k-m-\frac{\nu}{2}-\frac{1}{4} & -k+m+\frac{\nu}{2}+\frac{3}{4} \\ \frac{1}{4}+\frac{\nu}{2}+m\perp m, & \frac{1}{4}-\frac{\nu}{2}-m+m \end{matrix} \right]$$

$$= \frac{1}{2\nu}\, x^{\nu+\frac{1}{2}}\, f_{\nu,k,m}\left(\frac{x^2}{4}\right)$$

which plays the role of a kernel in another generalisation of Hankel transform (9, p. 271).

The object of this paper is to determine a condition, under which a function can be a solution of the integral equation (1.4). Here we will also give a proof of the converse, followed by investigation by means of it of some new self-reciprocal functions under various generalisations of Hankel transform.

2. THEOREM 1 : A necessary and sufficient condition, so that a function $f(x)$ of A (α, a) may be self-reciprocal in (1.4), is that

$$\text{(2.1)} \qquad f(x) = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \beta^{-s/2\gamma} \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)} \psi(s)\, x^{-s}\, ds,$$

where $\psi(s)$ is regular and satisfies the condition

$$\text{(2.2)} \qquad \psi(s) = \psi(1-s)$$

in the strip

$$\text{(2.3)} \qquad a < \sigma < 1-a$$

and

$$\text{(2.4)} \qquad \psi(s) = o\left(e^{\{(q-p)\frac{\pi}{4}-\alpha+\eta\}|t|}\right)$$

for every η and uniformly in any strip interior to (2.3) and c is any value of σ in (2.3).

PROOF : Let us now investigate the form of the function $f(x)_r$ which satisfies the integral equation

$$(2.5)\quad f(x)=2\gamma\beta^{\frac{1}{2r}}\int_0^\infty G^{q,p}_{2p,\,2q}\left[\beta^2(xy)^{2\gamma}\left|\begin{matrix}\frac{2\gamma-1}{4\gamma}+a_1,\ \ldots,\ \frac{2\gamma-1}{4\gamma}+a_p,\ \frac{2\gamma-1}{4\gamma}-a_1,\ \ldots,\ \frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\ \ldots,\ \frac{2\gamma-1}{4\gamma}+b_q,\ \frac{2\gamma-1}{4\gamma}-b_1,\ \ldots,\ \frac{2\gamma-1}{4\gamma}-b_q\end{matrix}\right.\right]\times f(y)\,dy.$$

or in other words the function $f(x)$ is self-reciprocal in the generalised Hankel transform (1.4).

Let $F(s)$ be the Mellin transform of $f(x)$, then

$$(2.6)\quad F(s)=\int_0^\infty x^{s-1}f(x)\,dx$$

$$(2.7)\quad =2\gamma\beta^{1/2\gamma}\int_0^\infty x^{s-1}\,dx$$

$$\times\int_0^\infty G^{q,p}_{2p,\,2q}\left[\beta^2(xy)^{2\gamma}\left|\begin{matrix}\frac{2\gamma-1}{4\gamma}+a_1,\ \ldots,\ \frac{2\gamma-1}{4\gamma}+a_p,\ \frac{2\gamma-1}{4\gamma}-a_1\ldots,\ \frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\ldots,\ \frac{2\gamma-1}{4\gamma}+b_q,\ \frac{2\gamma-1}{4\gamma}-b_1,\ \ldots,\ \frac{2\gamma-1}{4\gamma}-b_q\end{matrix}\right.\right]f(y)\,dy$$

$$=\beta^{1/2\gamma-s/\gamma}\int_0^\infty y^{-s}f(y)\,dy$$

$$\times \int_0^\infty x^{s/2\gamma-1}\; G^{q,\,p}_{2p,\,2q}$$

$$\left[x \left|\begin{matrix}\frac{2\gamma-1}{4\gamma}+a_1, \ldots \frac{2\gamma-1}{4\gamma}+a_p, \frac{2\gamma-1}{4\gamma}-a_1, \ldots, \frac{2\gamma-1}{4\gamma}-a_p \\ \frac{2\gamma-1}{4\gamma}+b_1, \ldots, \frac{2\gamma-1}{4\gamma}+b_q, \frac{2\gamma-1}{4\gamma}-b_1, \ldots, \frac{2\gamma-1}{4\gamma}-b_q\end{matrix}\right.\right] dx,$$

on replacing $\beta^2\ (xy)^{2\gamma}$ by x and changing the order of integration.

Hence

$$\text{(2.8)}\qquad F(s) = \frac{\beta^{\frac{1}{2\gamma}-\frac{s}{\gamma}} \prod_{j=1}^{\gamma} \Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j-\frac{s}{2\gamma}\right)}{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma+1}{4\gamma}+b_j-\frac{s}{2\gamma}\right) \prod_{i=1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)} \times F(1-s).$$

The inversion of the order of integration in (2.7) can easily be justified by de La Vallee' Poussin's theorem (4, p. 504), if the integral, involved in (1.4), is absolutely convergent and Mellin transform of $|f(x)|$ exists.

If now we suppose that

$$\text{(2.9)}\quad F(s) = \beta^{-s/2\gamma} \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right)}{\prod_{j+1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma}+a_j+\frac{s}{2\gamma}\right)} \psi(s),$$

we see that $\psi(s)$ satisfies the functional relation (2.2) and therefore by applying Mellin's inversion formula (13, p. 7) to (2.9) we obtain

$$(2.10)\quad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \beta^{-s/2\gamma} \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)} \psi(s)\, x^{-s}\, ds.$$

The further proof follows as in the corresponding theorem of Hankel transform (13, p. 232).

3. ILLUSTRATION : The Mellin transform of function

$$(3.1)\quad G^{l+q,l}_{r+p,r+q}\left[x\,\middle|\,\begin{matrix}\frac{2\gamma-1}{4\gamma}+\alpha_1,\dots,\frac{2\gamma-1}{4\gamma}+\alpha_r,\ \frac{2\gamma-1}{4\gamma}\quad \alpha_1,\dots,\ \frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\dots,\frac{2\gamma-1}{4\gamma}+b_q,\ \frac{2\gamma-1}{4\gamma}+\beta_1,\dots,\frac{2\gamma-1}{4\gamma}+\beta_r\end{matrix}\right],$$

$$O \leq 2l \leq 2r < 4l+q-p,$$

is
$$\frac{\prod_{j=1}^{q}\Gamma\left(\frac{2r-1}{4r}+b_j+s\right)\prod_{j=1}^{l}\Gamma\left(\frac{2\gamma-1}{4\gamma}+\beta_j+s\right)\prod_{j=1}^{l}\Gamma\left(\frac{2\gamma+1}{4\gamma}-\alpha_j-s\right)}{\prod_{j=1}^{p}\Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+s\right)\prod_{j=l+1}^{r}\Gamma\left(\frac{2\gamma+1}{4\gamma}-\beta_j-s\right)\prod_{j=l+1}^{r}\Gamma\left(\frac{2\gamma-1}{4\gamma}+\alpha_j+s\right)}.$$

Hence using Mellin's inversion formula and replacing x by $\beta x 2^{\gamma}$ and s by $\frac{s}{2\gamma}$ we have that

$$(3.2)\quad 2\gamma\ G^{l+q,\,l}_{r+p,\,r+q}\left[\beta x^{2\gamma}\,\middle|\,\begin{matrix}\frac{2\gamma-1}{4\gamma}+\alpha_1,\dots,\frac{2\gamma-1}{4\gamma}+\alpha_r,\ \frac{2\gamma-1}{4\gamma}-a_1,\dots,\frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\dots,\frac{2\gamma-1}{4\gamma}+b_q,\ \frac{2\gamma-1}{4\gamma}+\beta_1,\dots,\frac{2\gamma-1}{4\gamma}+\beta_q\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \beta^{-\frac{s}{2\gamma}} \frac{\prod_{j=1}^{q}\Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{l}\Gamma\left(\frac{2\gamma-1}{4\gamma}+\beta_j+\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p}\Gamma\left(\frac{2\gamma+1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)\prod_{j=l+1}^{r}\Gamma\left(\frac{2\gamma+1}{4\gamma}-\beta_j-\frac{s}{2\gamma}\right)}\times$$

$$\frac{\prod_{j=1}^{l}\Gamma\left(\frac{2\gamma+1}{4\gamma}-\alpha_j-\frac{s}{2\gamma}\right)}{\prod_{j=l+1}^{r}\Gamma\left(\frac{2\gamma-1}{4\gamma}+\alpha_j+\frac{s}{2\gamma}\right)}x^{-s}\,ds.$$

Thus in (3·2) the right hand side is of the same form as that of (2.1) with

$$\psi(s)=\frac{\prod_{j=1}^{l}\Gamma\left(\frac{2\gamma-1}{4\gamma}+\beta_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{l}\Gamma\left(\frac{2\gamma+1}{4\gamma}-\alpha_j-\frac{s}{2\gamma}\right)}{\prod_{j=l+1}^{r}\Gamma\left(\frac{2\gamma+1}{4\gamma}-\beta_j-\frac{s}{2\gamma}\right)\prod_{j=l+1}^{r}\Gamma\left(\frac{2\gamma-1}{4\gamma}+\alpha_j+\frac{s}{2\gamma}\right)},$$

which satisfies the functional relation (2.2) if

$$\alpha_j+\beta_j=0,\ j=1,\ldots,r.$$

Therefore we find that

$$(3.3)\qquad 2\gamma G^{l+q,\,l}_{r+p,\,r+q}\left[\beta x^{2\gamma}\,\middle|\,\begin{matrix}\frac{2\gamma-1}{4\gamma}+\alpha_1,\ldots,\frac{2\gamma-1}{4\gamma}+\alpha_r,\frac{2\gamma-1}{4\gamma}-a_1,\ldots,\frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\ldots,\frac{2\gamma-1}{4\gamma}+b_q,\frac{2\gamma-1}{4\gamma}-\alpha_1,\ldots,\frac{2\gamma-1}{4\gamma}-\alpha_r\end{matrix}\right]$$

is self-reciprocal in (1.4) provided that $0\leqslant l\leqslant r<2l-\frac{1}{2}(p-q)$,

$$\frac{1-2\gamma}{2}-2\gamma\min_{1\leqslant j\leqslant q}\mathrm{Re}(b_j)<\mathrm{Re}(s)<\frac{1+2\gamma}{2}-2\gamma\max_{1\leqslant j\leqslant l}\mathrm{Re}(\alpha_j),$$

and $\frac{1-2\gamma}{2}+2\gamma\min\limits_{1\leqslant j\leqslant l}\mathrm{Re}(\alpha_j)<\mathrm{Re}(s)<\frac{1+2\gamma}{2}-2\gamma\max\limits_{1\leqslant j\leqslant l}\mathrm{Re}(\alpha_j)$

Many known and unknown self-reciprocal functions can be derived as particular cases of (3.3) under various generalisations, given in section 1. For example

(i) With $\gamma=1$, $\beta=\frac{1}{2}$ and replacing α_j by $\alpha_j-\frac{1}{4}$, $j=1,\ldots\ldots r$; a_j by $a_j-\frac{1}{4}$, $j=1, \ldots\ldots, p$ and b_j by $b_j-\frac{1}{4}$, $j-1, \ldots, q$, we obtain the function (12, p. 118), which is self-reciprocal in the transform with the kernel (1·2),

and (ii) when $\gamma=1$ $\beta=\frac{1}{2}$, $p=o$, $q=1$ and $b_1-\frac{1}{4}+\frac{v}{2}$ we have the function (9, p. 286), self-reciprocal in Hankel transform.

4. THEOREM 2 : If the function F(s), defined by (2·6), (wherein the function $f(x)$ does not contain the parameter (s) satisfies the functional equation (2·8), then the function $f(x)$ is defined by (2·5), i.e. $f(x)$ is self-reciprocal in (1·4).

PROOF : By using the known integral (2·7), obtained in the earlier section 2, for the fraction involving Gamma-functions, (2·8) may be written as

$$F(s)=2\beta^{1/2\gamma}\int_0^\infty f(y)\left[\frac{\beta^{-s/r}\prod_{j-1}^{q}\Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right)\prod_{j-1}^{p}}{2\gamma\prod_{j-1}^{q}\Gamma\left(\frac{2\gamma+1}{4\gamma}+b_t-\frac{s}{2r}\right)\prod_{j-1}^{p}}\right.$$

$$\left.\frac{\Gamma\left(\frac{2\gamma+1}{4\gamma}-a_j-\frac{s}{2\gamma}\right)y^{-s}}{\Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)}\right]dy$$

$$(4\cdot1)\qquad =2\gamma\beta^{1/2\gamma}\int_0^\infty f(y)\,dy\int_0^\infty x^{s-1}$$

$$G^{q,\,p}_{2p,\,2q}\left[\beta^2(xy)^{2\gamma}\,\middle|\,\begin{matrix}\frac{2\gamma-1}{\gamma}+a_1\ldots,\frac{2\gamma-1}{\gamma4}+a_p,\frac{2\gamma-1}{4\gamma}-a_1,\ldots,\frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\ldots,\frac{2\gamma-1}{4\gamma}+b_q,\frac{2\gamma-1}{4\gamma}-b_1,\ldots\frac{2\gamma-1}{4\gamma}-b_q\end{matrix}\right]dx$$

$$
-\int_0^\infty x^{s-1}\,dx \int_0^\infty 2\gamma\beta^{1/2\gamma}
$$

$$
G^{q,\,p}_{2p,\,2q}\left[\beta^2 (xy)^{2\gamma}\left|\begin{matrix}\frac{2\gamma-1}{4\gamma}+a_1,\ldots,\frac{2\gamma-1}{4\gamma}+a_p,\ \frac{2\gamma-1}{4\gamma}-a_1,\ldots,\frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\ldots,\frac{2\gamma-1}{4\gamma}+b_q,\ \frac{2\gamma-1}{4\gamma}-b_1,\ldots,\frac{2\gamma-1}{4\gamma}-b_q\end{matrix}\right.\right] f(y)dy
$$

on changing the order of integration.

By using a known result (6, p. 3) the integrals, defining F (s) and F $(1-s)$, are convergent, if

$$
f(x) = 0\left(x^{\,|\mathrm{Re}(s-\frac{1}{2})|-\frac{1}{2}}\right) \quad \text{as } x\to 0 \quad \text{and}
$$

$$
f(x) = 0\left(x^{-|\mathrm{Re}(s-\frac{1}{2})|-\frac{1}{2}}\right) \quad \text{as } x\to\infty\ .
$$

Therefore in view of the asymptotic behaviour of Meijer's G-function (1, p. 212) we see that

(i) the x-integral in (4·1) is absolutely convergent if

$$
\frac{1-2\gamma}{2}-2\gamma \min_{1\leqslant j\leqslant q}\mathrm{Re}\,(b_j)<\mathrm{Re}\,(s)<\frac{1+2\gamma}{2}-2\gamma \max_{1\leqslant j\leqslant p}\mathrm{Re}\,(a_j)
$$

and (ii) the y—integral is so, when

$$
\gamma+2\gamma \min_{1\leqslant j\leqslant q}\mathrm{Re}\,(b_j)+|\,\mathrm{Re}\,(s-\tfrac{1}{2})\,|>0 \quad \text{and}
$$

$$
\gamma-2\gamma \max_{1\leqslant j\leqslant p}\mathrm{Re}\,(a_j)+|\,\mathrm{Re}\,(s-\tfrac{1}{2})\,|>0\ .
$$

Hence the change of order of integration in (4·1) is permissible by de La Vallee' Poussin's theorem,

Then

$$\int_0^\infty x^{s-1} f(x)\,dx \equiv F(s) = \int_0^\infty x^{s-1}\,dx$$

$$\times \int_0^\infty 2\gamma\beta^{\frac{1}{2\gamma}} G^{q,\,p}_{2p,\,2q}\left[\beta^2 (xy)^{2\gamma}\left|\begin{matrix}\frac{2\gamma-1}{4\gamma}+a_1,\ldots,\frac{2\gamma-1}{4\gamma}+a_p,\ \frac{2\gamma-1}{4\gamma}-a_1,\ldots,\frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\ldots,\frac{2\gamma-1}{4\gamma}+b_q,\ \frac{2\gamma-1}{4\gamma}-b_1,\ldots,\frac{2\gamma-1}{4\gamma}-b_q\end{matrix}\right.\right]$$

$$\times f(y)\,dy$$

or

$$(4{\cdot}2)\qquad \int_0^\infty x^{s-1}\Big\{ f(x)-2\gamma\beta^{\frac{1}{2\gamma}}$$

$$\int_0^\infty G^{q,\,p}_{2p\,2q}\left[\beta^2(xy)^{2\gamma}\left|\begin{matrix}\frac{2\gamma-1}{4\gamma}+a_1,\ldots,\frac{2\gamma-1}{4\gamma}+a_p,\ \frac{2\gamma-1}{4\gamma}-a_1,\ldots,\frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\ldots,\frac{2\gamma-1}{4\gamma}+b_q,\ \frac{2\gamma-1}{4\gamma}-b_1,\ldots,\frac{2\gamma-1}{4\gamma}-b_q\end{matrix}\right.\right]$$

$$\times f(y)\,dy\Big\}\,dx = 0\,.$$

Since (4·2) holds for general values of the parameter s, which occurs in the integrand only as exponent of x (with respect to which the integration is performed), it follows by an obvious modification of Lerch's theorem (7, p. 340) that the integrand is identically zero, that is

$$f(x) = 2\gamma\beta^{\frac{1}{2\gamma}} \int_0^\infty G^{q,\,p}_{2p,\,2q}$$

$$\left[\beta^2 (xy)^{2\gamma}\left|\begin{matrix}\frac{2\gamma-1}{4\gamma}+a_1,\ldots,\frac{2\gamma-1}{4\gamma}+a_p,\ \frac{2\gamma-1}{4\gamma}-a_1,\ldots,\frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\ldots,\frac{2\gamma-1}{4\gamma}+b_q,\ \frac{2\gamma-1}{4\gamma}-b_1,\ldots,\frac{2\gamma-1}{4\gamma}-b_q\end{matrix}\right.\right] f(y)\,dy.$$

whence the proposition.

4·1. PARTICULAR CASES : Taking $p=1$, $q=2$, $\gamma=1$, $\beta=\frac{1}{2}$, $a_1=k-m-\frac{1}{2}-\frac{\nu}{2}$, $b_1=\frac{\nu}{2}$ and $b_2=\frac{\nu}{2}+2m$, we shall arrive at the case due to R. K. Saxena (11, p. 126), where as in addition to the above substitutions having $k+m=\frac{1}{2}$, we obtain H. C. Gupta's case (6, p. 4).

5. Consider the function

$$f(x) = G^{\sigma,\tau}_{m,n}\left[\left(\frac{x}{\alpha}\right)^{\rho} \middle| \begin{matrix}\alpha_1, \dots, \alpha_m \\ \beta_1, \dots, \beta_n\end{matrix}\right], \qquad 0 \leqslant \sigma \leqslant n,\ 0 \leqslant \tau \leqslant m.$$

Its Mellin transform (2, p. 337) is

$$(5.1)\quad F(s) = \frac{l}{\rho}\,\frac{\prod_{j=1}^{\sigma}\Gamma\left(\beta_j+\frac{s}{\rho}\right)\prod_{j=1}^{\tau}\Gamma\left(1-\alpha_j-\frac{s}{\rho}\right)}{\prod_{j=\sigma+1}^{n}\Gamma\left(1-\beta_j-\frac{s}{\rho}\right)\prod_{j=\tau+1}^{m}\Gamma\left(\alpha_j+\frac{s}{\rho}\right)}\,\alpha^{s}.$$

If F (s) satisfies the functional equation (2·8), then equating the values of $\frac{F(s)}{F(1-s)}$ from both the equations (2·8) and (5·1) we get

$$(5\cdot2)\quad \alpha^{2s-1}\,\frac{\prod_{j=1}^{\sigma}\Gamma\left(\beta_j+\frac{s}{\rho}\right)\prod_{j=\sigma+1}^{n}\Gamma\left(1-\frac{1}{\rho}-\beta_j+\frac{s}{\rho}\right)\prod_{j=1}^{\tau}\Gamma\left(1-\alpha_j-\frac{s}{\rho}\right)}{\prod_{j=1}^{\sigma}\Gamma\left(\frac{1}{\rho}+\beta_j-\frac{s}{\rho}\right)\prod_{j=\sigma+1}^{n}\Gamma\left(1-\beta_j-\frac{s}{\rho}\right)\prod_{j=1}^{\tau}\Gamma\left(1-\frac{1}{\rho}-\alpha_j+\frac{s}{\rho}\right)}$$

$$\frac{\prod_{j=\tau+1}^{m}\Gamma\left(\frac{1}{\rho}+\alpha_j-\frac{s}{\rho}\right)}{\prod_{j=\tau+1}^{m}\Gamma\left(\alpha_j+\frac{s}{\rho}\right)}$$

$$= \beta^{\frac{1}{2\gamma}-\frac{s}{\gamma}}\,\frac{\prod_{j=1}^{q}\Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{p}\Gamma\left(\frac{2\gamma+1}{4\gamma}-a_j-\frac{s}{2\gamma}\right)}{\prod_{j=1}^{q}\Gamma\left(\frac{2\gamma+1}{4\gamma}+b_j-\frac{s}{2\gamma}\right)\prod_{j=1}^{p}\Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)}$$

Applying Gauss' multiplication theorem to the right hand side of (5·2) we have

$$(5\cdot 3)\quad \alpha^{2s-1}\ \frac{\prod_{j=1}^{\sigma}\Gamma\left(\beta_j+\frac{s}{\rho}\right)\prod_{j=\sigma+1}^{n}\Gamma\left(1-\frac{1}{\rho}-\beta_j+\frac{s}{\rho}\right)\prod_{j=1}^{\tau}\Gamma\left(1-\alpha_j-\frac{s}{\rho}\right)}{\prod_{j=1}^{\sigma}\Gamma\left(\frac{1}{\rho}+\beta_j-\frac{s}{\rho}\right)\prod_{j=\sigma+1}^{n}\Gamma\left(1-\beta_j-\frac{s}{\rho}\right)\prod_{j=1}^{\tau}\Gamma\left(1-\frac{1}{\rho}-\alpha_j+\frac{s}{\rho}\right)}$$

$$\frac{\prod_{j=\tau+1}^{m}\Gamma\left(\frac{1}{\rho}+\alpha_j-\frac{s}{\rho}\right)}{\prod_{j=\tau+1}^{m}\Gamma\left(\alpha_j+\frac{s}{\rho}\right)}$$

$$=\beta^{\frac{1}{2\gamma}-\frac{s}{\gamma}}\ K^{(p-q)\left(\frac{1}{2\gamma}-\frac{s}{\gamma}\right)}\ \frac{\prod_{j=1}^{q}\left[\prod_{i=1}^{k}\Gamma\left(\frac{2\gamma-1}{4\gamma k}+\frac{b_j}{k}+\frac{s}{2\gamma k}+\frac{i-1}{k}\right)\right]}{\prod_{j=1}^{q}\left[\prod_{i=1}^{k}\Gamma\left(\frac{2\gamma+1}{4\gamma k}+\frac{b_j}{k}-\frac{s}{2\gamma k}+\frac{i-1}{k}\right)\right]}$$

$$\frac{\prod_{j=1}^{p}\left[\prod_{i=1}^{k}\Gamma\left(\frac{2\gamma+1}{4\gamma k}-\frac{a_j}{k}-\frac{s}{2\gamma k}+\frac{i-1}{k}\right)\right]}{\prod_{j=1}^{p}\left[\prod_{i=1}^{k}\Gamma\left(\frac{2\gamma-1}{4\gamma k}-\frac{a_j}{k}+\frac{s}{2\gamma k}+\frac{i-1}{k}\right)\right]}.$$

where k is a positive integer.

This will admit several solutions, of which only a few are mentioned here as particular cases.

To secure the equality of parameters in (5·3) we see that, if

(A) $\quad k=1,\ \alpha=\beta^{-1/2\gamma},\ \rho=2\gamma,\ \sigma=q,\ \tau=0,\ m=p,\ n=q,\ \alpha_j=\frac{2\gamma-1}{4\gamma}-a_j,$

$j=1,\ldots,p$ and $\beta_j=\frac{2\gamma-1}{4\gamma}+b_j,\ j=1,\ldots,q$, we obtain

$$(5{\cdot}4)\qquad G^{q,\,o}_{p,\,q}\left[\beta x^{2\gamma}\,\middle|\,\begin{matrix}\frac{2\gamma-1}{4\gamma}-a_1,\ldots,\frac{2\gamma-1}{4\gamma}-a_p\\ \frac{2\gamma-1}{4\gamma}+b_1,\ldots,\frac{2\gamma-1}{4\gamma}+b_r\end{matrix}\right],$$

as a self-reciprocal function in (1·4), provided that $q > p$ and $\frac{1-2\gamma}{2} = 2\gamma \min\limits_{1\leqslant j\leqslant q}.$ Re (b_j) $<$ Re (s).

(B) $k=2$, $\alpha=\beta^{-\frac{1}{2\gamma}}\,2^{\frac{q-p}{2\gamma}}$, $\rho=4\gamma$, $\sigma=q$, $\tau=p$, $m=2p$, $n=2q$ and choosing the other parameters suitably by different combinations of Gamma-functions on both the sides of (5·3) we find that

(i) the function

$$(5{\cdot}5)\qquad G^{q,\,p}_{2p,\,2q}\left[4^{p-q}\,\beta^2 x^{4\gamma}\,\middle|\,\begin{matrix}\frac{2\gamma-1}{8\gamma}+\frac{a_1}{2},\ldots,\frac{2\gamma-1}{8\gamma}+\frac{a_p}{2},\ \frac{2\gamma-1}{8\gamma}-\frac{a_1}{2},\ldots,\frac{2\gamma-1}{8\gamma}-\frac{a_p}{2}\\ \frac{2\gamma-1}{8\gamma}+\frac{b_1}{2},\ldots,\frac{2\gamma-1}{8\gamma}+\frac{b_q}{2},\ \frac{2\gamma-1}{8\gamma}-\frac{b_1}{2},\ldots,\frac{2\gamma-1}{8\gamma}-\frac{b_q}{2}\end{matrix}\right]$$

is self-reciprocal in (1.4), provided that

$$\frac{1-2\gamma}{2}-2\gamma\min_{1\leqslant j\leqslant q}.\ \mathrm{Re}\,(b_j) < \mathrm{Re}\ (s) < \frac{1+6\gamma}{2}-2\gamma\max_{1\leqslant j\leqslant p}.\ \mathrm{Re}\,(a_j),$$

(ii) the function

$$(5.6)\qquad G^{q,\,p}_{2p,\,2q}\left[4^{p-q}\,\beta^2 x^{4\gamma}\,\middle|\,\begin{matrix}\frac{6\gamma-1}{8\gamma}+\frac{a_1}{2},\ldots,\frac{6\gamma-1}{8\gamma}+\frac{a_p}{2},\ \frac{6\gamma-1}{8\gamma}-\frac{a_1}{2},\ldots,\frac{6\gamma-1}{8\gamma}-\frac{a_p}{2}\\ \frac{2\gamma-1}{8\gamma}+\frac{b_1}{2},\ldots,\frac{2\gamma-1}{8\gamma}+\frac{b_q}{2},\ \frac{2\gamma-1}{8\gamma}-\frac{b_1}{2},\ldots,\frac{2\gamma-1}{8\gamma}-\frac{b_q}{2}\end{matrix}\right]$$

is also self-reciprocal in (1.4), provided that

$$\frac{1-2\gamma}{2}-2\gamma \min_{1\leq j\leq q} \operatorname{Re}(b_j) < \operatorname{Re}(s) < \frac{1+2\gamma}{2} - 2\gamma \max_{1\leq j\leq p} \operatorname{Re}(a_j),$$

(iii) the function

$$(5.7) \qquad G^{q,\,p}_{2p,\,2q}\left[4^{p-q}\beta^4 x^{4\gamma} \,\middle|\, \begin{matrix} \frac{2\gamma-1}{8\gamma}+\frac{a_1}{2}, \ldots, \frac{2\gamma-1}{8\gamma}+\frac{a_p}{2}, \frac{2\gamma-1}{8\gamma}-\frac{a_1}{2}, \ldots, \frac{2\gamma-1}{8\gamma}-\frac{a_p}{2} \\ \frac{6\gamma-1}{8\gamma}+\frac{b_1}{2}, \ldots, \frac{6\gamma-1}{8\gamma}+\frac{b_q}{2}, \frac{6\gamma-1}{8\gamma}-\frac{b_1}{2}, \ldots, \frac{6\gamma-1}{8\gamma}-\frac{b_q}{2} \end{matrix}\right]$$

is a self-reciprocal function in (1.4), provided that

$$\frac{1-6\gamma}{2}-2\gamma \min_{1\leq j\leq q} \operatorname{Re}(b_j) < \operatorname{Re}(s) < \frac{1+6\gamma}{2} - 2\gamma \max_{1\leq j\leq p} \operatorname{Re}(a_j)$$

and

(iv) the function

$$(5.8)\ G^{q,p}_{2p,\,2q}\left[4^{p-q}\beta^4 x^{4\gamma} \,\middle|\, \begin{matrix} \frac{6\gamma-1}{8\gamma}+\frac{a_1}{2}, \ldots, \frac{6\gamma-1}{8\gamma}+\frac{a_p}{2}, \frac{6\gamma-1}{8\gamma}-\frac{a_1}{2}, \ldots, \frac{6\gamma-1}{8\gamma}-\frac{a_p}{2} \\ \frac{6\gamma-1}{8\gamma}+\frac{b_1}{2}, \ldots, \frac{6\gamma-1}{8\gamma}+\frac{b_q}{2}, \frac{6\gamma-1}{8\gamma}-\frac{b_1}{2}, \ldots, \frac{6\gamma-1}{8\gamma}-\frac{b_q}{2} \end{matrix}\right]$$

is another self-reciprocal function in (1.4), provided that

$$\frac{1-6\gamma}{2}-2\gamma \min_{1\leq j\leq q} \operatorname{Re}(b_j) < \operatorname{Re}(s) < \frac{1+2\gamma}{2} - 2\gamma \max_{1\leq j\leq p} \operatorname{Re}(a_j).$$

The functions (5.4) to (5.8) will yield, as particular cases, many self-reciprocal functions under various generalisations of the Hankel transform in view of section 1. As for example with $p=1$, $q=2$, $\gamma=1$, $\beta=\frac{1}{2}$ and setting parameters in (5.4) and (5.5) we obtain known cases (9, p. 287) and (11, p. 129) respectively.

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SOME POLYNOMIALS OF SHEFFER A–TYPE ZERO (I)

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ABSTRACT

Shrivastav (1964) has studied the polynomials $A_n^{(\alpha)}(x)$ defined by

$$A_n^{(\alpha)}(x) = \sum_{r=0}^{n} (-1)^r \binom{\alpha+r}{r} \frac{x^{n-r}}{(n-r)!}$$

Sylvester has studied the polynomials

$$\phi_n(x) = \sum_{r=0}^{n} (-1)^r \binom{-x}{r} \frac{x^{n-r}}{(n-r)!}$$

In this note the author has generalised $\phi_n(x)$ in the form

$$\phi_n^{(\alpha)}(x) = \sum_{r=0}^{n} (-1)^r \binom{-x-\alpha}{r} \frac{x^{n-r}}{(n-r)!}$$

and has obtained various recurrence relations, integrals and expansions involving $A_n^{(\alpha)}(x)$, $\phi_n(x)$ and $\phi_n^{(\alpha)}(x)$. The expansion of $A_n^{(\alpha)}(x)$ in terms of the modified Bessel function of the first kind with an arbitrary parameter λ seems to be worthy of note.

1. INTRODUCTION

Recently Shrivastav [15] studied the polynomials

$A_n^{(\alpha)}(x)$ satisfying the relation,

$$\begin{cases} \sum_{r=0}^{\infty} A_r^{(\alpha)}(x)\, L_{n-r}^{(\alpha+r)}(x) = 0, \; n \geq 1 \\ A_0^{(\alpha)}(x) = 1 \end{cases}$$

where $L_n^{(\alpha)}(x)$ is the generalised Laguerre polynomial.

He showed that :—

$$(1.1) \quad A_n^{(\alpha)}(x) = \sum_{r=0}^{n} (-1)^r \binom{\alpha+r}{r} \frac{x^{n-r}}{(n-r)!} = L_n^{(-\alpha-n-1)}(-x)$$

and

$$(1.2) \quad \sum_{n=0}^{\infty} t^n A_n^{(\alpha)}(x) = (1+t)^{-1-\alpha} \cdot e^{xt} .$$

Sylvester has studied the polynomials $\phi_n(x)$ [7, p. 255] defined as

$$(1.3) \quad \phi_n x = \sum_{r=0}^{n} (-1)^r \binom{-x}{r} \frac{x^{n-r}}{(n-r)!}$$

so that $\sum_{n=0}^{\infty} t^n \phi_n(x) = (1-t)^{-x} \cdot e^{x \cdot t}$

The generating function in (1.2) has earlier been used by Al-Salam [1] in connection with his study of the Bessel polynomials. Singh [16] has also studied some properties of $A_n^{(\alpha)}(x)$. The object of this note is to investigate further properties of $A_n^{(\alpha)}(x)$ and $\phi_n(x)$ and to generalise $\phi_n(x)$ in the form

$$(1.4)\quad \phi_n^{(\beta)}(x) = \sum_{r=0}^{n} (-1)^r \binom{-x-\beta}{r} \frac{x^{n-r}}{(n-r)!}$$

with generating function

$$(1.5)\quad \sum_{n=0}^{\infty} t^n \phi_n^{(\beta)}(x) = (1-t)^{-x-\beta} \cdot e^{xt}$$

In a recent paper Rangarajan [11] has studied some general properties of Appell polynomials of the type of $A_n^{(\alpha)}(x)$; but the results about $A_n^{(\alpha)}(x)$ in our paper are entirely different from those of his paper.

2. The following results will be required in the investigations that follow :—

$$(a)\quad R_n(a, x) = \frac{(a)_{2n}}{n!\,(a)_n}\ {}_1F_1\left[\begin{matrix} -n\,;\, x \\ a+n \end{matrix}\right]$$

where $R_n(a, x)$ is the pseudo-Laguerre polynomial defined by Shivley [14].

$$(b)\quad Y_n^{(\alpha)}(x) = {}_2F_0\left[-n, n+\alpha+1 \,;\, \frac{-x}{2}\right]$$

where $Y_n^{(\alpha)}(x)$ is the Bessel polynomial as considered by Al-Salam [1]

$$(c)\quad M_{k,m}(x) = x^{\frac{1}{2}+m}\, e^{-\frac{1}{2}x}\ {}_1F_1\left[\begin{matrix} \frac{1}{2}+m-k\,;\, x \\ 1+2m \end{matrix}\right]$$

where $M_{k,m}(x)$ is the Whittaker's function.

$$(d)\quad M_{k,m}(x) = -\frac{\Gamma(1+2m)}{2\pi i}\, e^{-\frac{1}{2}x}\, x^{\frac{1}{2}+m} \int_{\infty}^{(0+)} (-t)^{-1-2m} \left(1+\frac{x}{t}\right)^{-\frac{1}{2}-m+k} \cdot e^{-t}\, dt$$

where the path of integration starts at infinity on the real axis, encircles the origin in the positive direction, and returns to plus infinity on the contour $|t| > |x|$.

(e) (Slater's expansion) [17, p. 32]*

$$
{}_1F_1[a; b; x] = e^{\frac{1}{2}x}\,\Gamma(b-a-\tfrac{1}{2})\,(\tfrac{1}{4}x)^{a-b+\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n \frac{(b-a-\frac{1}{2}+n)\,(2b-2a-1)_n\,(b-2a)_n}{n!\,(b)_n}\, I_{b-a-\frac{1}{2}+n}(\tfrac{1}{2}x)
$$

$$
(f)\quad 1+\sum_{n=1}^{\infty} \frac{(k+r)^k\,\omega^k}{k!} = \frac{e^{rz}}{1-z}
$$

$$
1+\sum_{k=1}^{\infty} \frac{r\,(k+r)^{k-1}\,\omega^k}{k!} = e^{rz}
$$

where $w = z\,e^{-z}$ and $|\omega| < e^{-1}$.

For the proof of (f) see Bromwich [4, p. 160].

(g) ${}_1F_1\,[\,a\,;\,c\,;\,x+y\,]$

$$
= \sum_{r=0}^{\infty} \frac{(a)_r\,(c-a)_r\,(xy)^r}{r!\,(c+r-1)_r\,(c)_{2r}}\;{}_1F_1\left[\begin{matrix} a+r\,;\,x \\ c+2r \end{matrix}\right]\;{}_1F_1\left[\begin{matrix} a+r\,;\,y \\ c+2r \end{matrix}\right]
$$

${}_1F_1\,[\,a; c; x\,]\,.\;{}_1F_1\,[\,a; c; y\,]$

$$
= \sum_{r=0}^{\infty} \frac{(-1)^r\,(a)_r\,(c-a)_r\,(xy)^r}{r!\,(c)_r\,(c)_{2r}}\;{}_1F_1\left[\begin{matrix} a+r\,;\,x+y \\ c+2r \end{matrix}\right]
$$

For the proof of (g) see Burchnall and Chaundy [5].

* The factor $(b-a-\frac{1}{2}+n)$ is missing in Slater's Book and should be inserted.

We observe that $A_n^{(\alpha)}(x)$, $\phi_n(x)$ and $\phi_n^{(\beta)}(x)$ though not orthogonal are simple sets of polynomials with Boas and Buck [3] type of generating functions. Hence they are of Sheffer A-type zero. It may be remarked that $\phi_n(x)$ and $\phi_n^{(\beta)}(x)$ are not Appell polynomials. But $A_n^{(\alpha)}(x)$ is an Appell polynomial and hence satisfies the relation,

$$(2.1)\quad \frac{d\,A_n^{(\alpha)}(x)}{d\,x} = A_{n-1}^{(\alpha)}(x).$$

It is easy to see that

$$(h)\quad A_n^{(\alpha)}(x) = \frac{(-1)^n\,(1+\alpha)_n}{n!}\,e^{-\frac{1}{2}x}(-x)^{\frac{\alpha+n}{2}}\,M_{n\,\frac{\alpha}{2},\,\frac{\alpha-n-1}{2}}(-x)$$

$$(i)\quad A_n^{(\alpha)}(x) = R_n(-\alpha-2n, -x)$$

$$(j)\quad A_n^{(\alpha)}(x) = \frac{x^n}{n!}\,Y_n^{(\alpha-n)}\left(-\frac{2}{x}\right)$$

3. Recurrence Relations.

Using Sheffer's theory [10, p. 225] we may immediately deduce that

$$(3.1)\quad x\,\frac{d\,A_n^{(\alpha)}(x)}{d\,x} - n\,A_n^{(\alpha)}(x) = (1+\alpha)\sum_{k=0}^{n-1}(-1)^k A_{n-1-k}^{(\alpha)}(x)$$

$$(3.2)\quad \frac{d\,\phi_n(x)}{d\,x} - 2\,\phi_{n-1}(x) = \sum_{k=1}^{n-1}\frac{\phi_{n-1-k}(x)}{k+1}$$

$$(3.3) \quad n\,\phi_n(x) - 2x\,\phi_{n-1}(x) = \sum_{k=1}^{n-1} x\,\phi_{n-1-k}(x)$$

Following Sister Celine's technique [8] we obtain

$$(3.4) \quad x\,A_{n-2}^{(\alpha)}(x) + (x-\alpha-n)\,A_{n-1}^{(\alpha)}(x) - n\,A_n^{(\alpha)}(x) = 0$$

(3.4) with (2.1) shows that $A_n^{(\alpha)}(x)$ satisfies the differential equation.

$$(3.5) \quad x\,\frac{d^2 y}{d x^2} + (x-\alpha-n)\,\frac{d y}{d x} - n\,y = 0$$

Proceeding as in (3.4) we find that

$$(3.6) \quad (n+1)\,\phi_{n+1}(x) - (n+2x)\,\phi_n(x) + x\,\phi_{n-1}(x) = 0$$

The following relations follow from the relations

$$A_n^{(\alpha)}(x) = \frac{(-1)^n\,(1+\alpha)_n}{n!}\;{}_1F_1\left[\begin{matrix}-n\,;\,-x\\ -\alpha-n\end{matrix}\right] = \frac{x^n}{n!}\;{}_2F_0\left[\begin{matrix}-n\,,\,1+\alpha\,;\,\frac{1}{x}\\ -\end{matrix}\right]$$

and the contiguous relations of the hypergeometric functions :—

$$\text{(a)} \quad A_n^{(\alpha)}(x) = A_n^{(\alpha+1)}(x) + A_{n-1}^{(\alpha+1)}(x)$$

$$\text{(b)} \quad (x+n)\,A_n^{(\alpha)}(x) + (\alpha+1)\,A_{n-1}^{(\alpha+1)}(x) = x\,A_n^{(\alpha-1)}(x)$$

$$\text{(c)} \quad (\alpha+1)\,A_n^{(\alpha+1)}(x) + x\,A_n^{(\alpha-1)}(x) = (x+\alpha+n+1)\,A_n^{(\alpha)}(x)$$

$$\text{(d)} \quad n\,A_n^{(\alpha)}(x) = (x+n)\,A_{n-1}^{(\alpha+1)}(x) - (\alpha+2)\,A_{n-1}^{(\alpha+2)}(x)$$

$$\text{(e)} \quad (\alpha+1)\,A_n^{(\alpha+1)}(x) + x\,A_{n-1}^{(\alpha)}(x) = (\alpha+n+1)\,A_n^{(\alpha)}(x)$$

For $\operatorname{Re} \alpha \geqslant -1$ (4.3) can be taken as the definition of $A_n^{(\alpha)}(x)$ and used to derive various relations.

From (2, d) and (2, h) we have

$$(4.4) \quad A_n^{\alpha} = \frac{(-1)^{\alpha+n+1}}{n!\, 2\pi i} \Gamma(-\alpha) \int_{\infty}^{(0+)} t^{\alpha+n} \left(1 - \frac{x}{t}\right)^n e^{-t}\, dt$$

where $\alpha \neq 0, 1, 2, \ldots\ldots\ldots$

For those values of α for which (4,3) is not valid, (4.4) can be considered an analytical continuation of (4.3).

Again by term-by-term integration we derive for $\alpha \geqslant n-i$

$$(4.5) \quad \frac{x^n}{\Gamma(1+\alpha)} \int_0^{\infty} t^{\alpha-n}\, P_n^{(\alpha-n,\, \beta)} \left(1 - \frac{2t}{x}\right) e^{-t}\, dt = A_n^{(\alpha+\beta)}(x)$$

and for $\beta \geqslant n-1$

$$(4.6) \quad \frac{(-x)}{\Gamma(1+\beta)} \int_0^{\infty} t^{\beta-n}\, P_n^{(\alpha,\, \beta-n)} \left(\frac{2t}{x} - 1\right) e^{-t}\, dt = A_n^{(\alpha+\beta)}(x)$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial.

5. Expansions :—

Consider

$$n! \sum_{k=1}^{\infty} \frac{k^{k-n}\, \omega^k}{k!} (-1)^n \, \phi n(-k)$$

$$= \sum_{r=0}^{n} \sum_{k=r\geqslant 1}^{\infty} \binom{n}{r} \frac{k^{k-r}\,\omega^k}{(k-r)!}$$

$$= \sum_{k=1}^{\infty} \frac{k^k\,\omega^k}{k!} + \sum_{r=1}^{n} \sum_{k=r}^{\infty} \binom{n}{r} \frac{k^{k-r}\,\omega^k}{(k-r)!}$$

$$= -1 + \frac{1}{1-z} + \sum_{r=1}^{n} \binom{n}{r} \omega^r \sum_{k=0}^{\infty} \frac{(k+r)^k\,\omega^k}{k!}$$

Using (2,f) we have finally.

$$(5.1) \qquad n! \sum_{k=1}^{\infty} \frac{k^{n-n}\,\omega^k\,(-1)^n \quad \phi_n\,(-k)}{k!} = -1 + \frac{(1+z)^n}{1-z}$$

Put $z=2$ then since $|w| < e^{-1}$ we have

$$n! \sum_{k=1}^{\infty} \frac{k-n \quad 2^k \quad e^{-2k}}{k!} \frac{(-1)^{n-1}\phi_n\,(-k)}{} = z^n + 1$$

Similarly for $\beta \neq 0, -1, -2, \ldots\ldots\ldots -n$

$$n! \sum_{k=0}^{\infty} \frac{(-1)^n\,(n+\beta)^{k-n-1}\,\omega^k}{k!} \phi_n^{(\beta)}\,(-k-\beta)$$

$$= \sum_{r=0}^{n} \binom{n}{r} \sum_{k=r}^{\infty} \frac{(k+\beta)^{k-r-1}\,\omega^k}{(k-r)}$$

$$= \sum_{r=0}^{n} \binom{n}{r} \frac{\omega^r}{r+\beta} \sum_{k=0}^{\infty} \frac{(r+\beta)\,(k+r+\beta)^{k-1}\,\omega^k}{k!}$$

Using (2,f) we have

$$(5.2)\quad n!\sum_{k=0}^{\infty}\frac{(-1)^n\ (k+\beta)^{k-n-1}}{k!}\ \omega^k\ \phi_n^{(\beta)}(-k-\beta)$$

$$= e^{\beta z}\sum_{r=0}^{n}\binom{n}{r}\frac{z^r}{r+\beta}$$

By expanding $A_n^{(\alpha)}(x)$ and rearranging the resulting double series we **derive** the following results :—

$$(5.3)\quad n!\sum_{k=1}^{\infty}k^{-\beta-n}A_n^{(\alpha)}(k)=\sum_{r=1}^{n}(-1)^r\binom{n}{r}(1+\alpha)_r\ \zeta(\beta+r)$$

where $\beta>1$ and $\zeta(z)$ is the Riemann zeta function.

$$(5.4)\quad \sum_{n=0}^{\infty}(c)_n\ t^n\ A_n^{(\alpha)}(x)=(1-xt)^{-c}\ {}_2F_0\left[c,\ 1+\alpha;\ \frac{t}{xt-1}\right]$$

$$(5.5)\quad \sum_{n=0}^{\infty}\frac{(c)_n\ t^n\ A_n^{(\alpha)}(x)}{(1+\alpha)_n}=\sum_{r=0}^{\infty}\frac{(c)_r\ (xt)^r}{r!\,(1+\alpha)_r}\ {}_2F_1\left[\begin{matrix}c+r,\ 1+\alpha;\ -t\\ 1+\alpha-r\end{matrix}\right]$$

When $c=1+\alpha$ in (5.5) we obtain (1.2).

We have

$$(5.6)\quad \lim_{\beta\to\infty}P_n^{(-\alpha-n-1,\ \beta)}\left(1+\frac{2x}{\beta}\right)=\lim_{\beta\to\infty}\frac{(-\alpha-n)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n,\ \beta-\alpha;\ \frac{-x}{\beta}\\ -\alpha-n\end{matrix}\right]$$

$$= \frac{(-1)^n \ (1+\alpha)_n}{n!} \ {}_1F_1\left[\begin{matrix} -n; -x \\ -\alpha-n \end{matrix}\right]$$

$$= A_n^{(\alpha)}(x)$$

Consider $\displaystyle\sum_{k=0}^{\infty} \frac{(2t)^k \ P_k^{(-1-\alpha-n,\ n-k)}(x)}{(-\alpha-n)_k}$

$$= \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \sum_{r=0}^{k} \frac{(-k)_r \ (-\alpha)_r}{r! \ (-\alpha-n)_r} \left(\frac{1-x}{2}\right)^r$$

$$= \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(2t)^{r+k} \ (-1)^r \ (-\alpha)_r}{r! \ k! \ (-\alpha-n)_r} \left(\frac{1-x}{2}\right)^r$$

$$= e^{2t} \ {}_1F_1\left[\begin{matrix} -\alpha; (x-1)t \\ -\alpha-n \end{matrix}\right]$$

Applying Kummer's transformation this is equal to

$$e^{(1+x)t} \ {}_1F_1\left[\begin{matrix} -n; -(x-1)t \\ -\alpha-n \end{matrix}\right]$$

Thus finally we have

$$(5.7) \quad \frac{e^{-(1+x)t} \ (1+\alpha)_n \ (-1)^n}{n!} \sum_{k=0}^{\infty} \frac{(2t)^k \ P_k^{(-1-\alpha-n,\ n-k)}(x)}{(-\alpha-n)_k}$$

$$= A_n^{(\alpha)}\{(x-1)t\}$$

Following Carlitz [6] put

$$f_n^{(\beta,\gamma)}(x) = \frac{2^n}{(1+\beta+\gamma-n)_n} P_n^{(\beta-n,\,\gamma-n)}(x),$$

then proceeding as in (5.7) we find

$$(5.8) \quad \frac{(1+\alpha)_n}{} \frac{e^{\frac{t}{2}(x-1)} (-1)^n}{n!} \sum_{k=0}^{\infty} \left(\frac{-t}{2}\right)^k f_k^{(n,\alpha)}(x) = A_n^{(\alpha)}(t)$$

From the integral representation (4.3) and by interchanging the order of integration and summation we find

$$(5.9) \quad \sum_{k=0}^{\infty} (-\lambda)^n A_n^{(k)}(x) = \frac{(-1)^n}{(1+\lambda)^{n+1}} e^{-(1+\lambda)x}$$

$$(5.10) \quad \sum_{k=0}^{\infty} \frac{(-\lambda)^k A_n^{(k)}(x)}{k!} = \frac{1}{n!} \int_0^{\infty} (x-t)^n e^{-t} J_0(2\sqrt{\lambda t})\, dt$$

$$(5.11) \quad \sum_{k=0}^{\infty} (-1)^k \lambda^{2k} A_n^{(2k)}(x) = \frac{x^n}{n!} \int_0^{\infty} (x-t)^n e^{-t} \cos \lambda t \quad dt$$

Interchange of the order of summation and integration is permissible for $k \geqslant 0$.

Following Al-Salam and Carlitz [2] we have the Turán expression

$$(5.12) \quad \left\{ A_n^{(\alpha)}(x) \right\}^2 - A_{n-1}^{(\alpha)}(x)\, A_{n+1}^{(\alpha)}(x)$$

$$= \frac{1}{(n+1)!} \sum_{r=0}^{n} \binom{-\alpha-1}{r} (n-r)!\, r! \left\{ A_{n-r}^{\alpha+r}(x) \right\}^2$$

which is positive for $\alpha \leqslant -n$.

Using (2, g) we derive

$$(5.13)\quad \frac{(-1)^n\, n!}{(1+\alpha)_n} A_n^{(\alpha)}(x+y)$$

$$= \sum_{r=0}^{n} \frac{(-n)_r\,(-\alpha)_r\,(x\,y)^r}{r!\,(-\alpha-n-1+r)_r\,(-\alpha-n)_{2r}} \left\{ \frac{(n-r)!}{(1+\alpha-r)_{n-r}} \right\}^2$$

$$A_{n-r}^{(\alpha-r)}(x)\; A_{n-r}^{(\alpha-r)}(y)$$

and

$$(5.14)\quad \left\{ \frac{n!}{(1+\alpha)_n} \right\}^2 A_n^{(\alpha)}(x)\; A_n^{(\alpha)}(y)$$

$$= \sum_{r=0}^{n} \frac{(-1)^n\,(-n)_r\,(-\alpha)_r\,(n-r)!\,(x\,y)^r}{r!\,(-\alpha-n)_r\,(-\alpha-n)_{2r}\,(1+\alpha-r)_{n-r}} A_{n-r}^{(\alpha-r)}(x+y)$$

From (2, e) we obtain

$$(5.15)\quad A_n^{(\alpha)}(x) = \frac{(-1)^{n+\alpha+\frac{1}{2}}\; e^{-\frac{1}{2}x}\,(\tfrac{1}{2}x)^{\alpha+\frac{1}{2}}\,(1+\alpha)_n\;\Gamma(-\alpha-\tfrac{1}{2})}{n!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-\alpha-\frac{1}{2}+k)\,(-2\alpha-1)_k\,(n-\alpha)_k\,(-1)^k}{k!\,(-\alpha-n)_k} I_{-\alpha-\frac{1}{2}+k}(-\tfrac{1}{2}x)$$

Now in the integral representation (4.4) put

$$\left(1-\frac{x}{t}\right)^n = \left(1-\frac{x}{t}\right)^{\alpha+\lambda+\frac{1}{2}+n} \sum_{r=0}^{\infty} \frac{(\alpha+\lambda+\frac{1}{2})_r}{r!} \left(\frac{x}{t}\right)^r,$$

$$\left|\frac{x}{t}\right| < 1$$

Hence $\frac{1}{2\pi i}\int_{\infty}^{(o+)} t^{\alpha+n}\left(1-\frac{x}{t}\right)^n e^{-t}\, dt$

$$= \sum_{r=0}^{\infty} \frac{(\alpha+\lambda+\frac{1}{2})_r \quad (x)^r \quad \Gamma(3/2+\alpha+\lambda+n)}{r\,!\;(-1)^{\alpha+n-r+1}\;\Gamma(\lambda+r+\frac{1}{2})} \quad A_{\alpha+\lambda+\frac{1}{2}+n}^{(-\lambda-r-\frac{1}{2})}(x)$$

and

$$(5.16)\quad A_n^{(\alpha)}(x) = \frac{\Gamma(-\alpha)\,\Gamma(3/\;+\alpha+\lambda+n)}{n\,!}\sum_{r=0}^{\infty}\frac{(\alpha+\lambda+\frac{1}{2})_r\quad(-x)^r}{r\,!\;\Gamma(\lambda+r+\frac{1}{2})}$$

$$A_{\alpha+\lambda+\frac{1}{2}+n}^{(-\lambda-r-\frac{1}{2})}(x)$$

Expressing $A_{\alpha+\lambda+\frac{1}{2}+n}^{(-\lambda-r-\frac{1}{2})}(x)$ in (5.16) in a series of Bessel functions as in (5.15) and rearranging the double series we finally obtain

$$(5.17)\quad A_n^{(\alpha)}(x) = \frac{(-1)^{n+\alpha+\frac{1}{2}}\quad e^{-\frac{1}{2}x}\;(\frac{1}{4}\,x^{\;-\lambda}\quad(1+\alpha)_n\;\;\Gamma(\lambda)}{n\,!}$$

$$\frac{\Gamma(1+\alpha)\,\Gamma(-\alpha)}{\Gamma(\frac{1}{2}-\lambda)\;\Gamma(\frac{1}{2}+\lambda)}$$

$$\times\sum_{k=0}^{\infty}\frac{(\lambda+k)\quad(2\,\lambda)_k\quad(\alpha+n+2\,\lambda+{}^1)_k}{k\,!\quad(-\alpha-n)_k}$$

$${}_3F_2\left[\begin{matrix}-k,\ \alpha+\lambda+\frac{1}{2},\ 2\,\lambda+k;\ 1\\ \lambda+\frac{1}{2},\ \alpha+n+2\,\lambda+1\end{matrix}\right]\mathrm{I}_{\lambda+k}\left(-\tfrac{1}{2}x\right)$$

When $\lambda = -\alpha-\frac{1}{2}$ we obtain (5.16)

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ON ZEROS OF A TRANSCENDENTAL FUNCTION ASSOCIATED WITH BESSEL FUNCTIONS OF THE FIRST KIND OF ORDERS ν AND $\nu+1$—PART II

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ABSTRACT

In recent years specific work has been done by several authors to develop the theory of zeros of the Bessel function $J_\nu(z)$ where ν and z are not necessarily real.

The results concerning chiefly with the zeros $j_{\nu, n}$ of the function $J_\nu(z)$ are

$$\text{I.} \quad J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j^2_{\nu,n}}\right)$$

$$\text{II.} \quad \sigma^{(1)}_{\nu,1} = 1/2^2(\nu+1), \quad \sigma^{(2)}_{\nu,1} = 1/2^4(\nu+1)^2(\nu+2), \ \ldots\ldots\ldots\text{etc},$$

$$\text{and III.} \quad \sigma^{(r)}_{\frac{1}{2},1} = \frac{2^{2r-1} B_r}{\lfloor 2r},$$

where B_r is the rth Bernoullian number.

The present paper embodies extensions of the results I, II and III to the zeros of the transcendental function

$$G_\nu(z) \equiv \{f(z)+\nu\} J_\nu(z) - z J_{\nu+1}(z)$$

under the cases

(*a*) $f(z)$ is an even function of z

and (*b*) $f(z)$ is an odd function of z.

1. *Introduction*: In one of our previous publications[1] we illustrated the method of determining the zeros of the transcendental function

$$F_\nu^{(h)}(z) = (h+\nu)\,J_\nu(z) - z\,J_{\nu+1}(z) \qquad (1.1)$$

where $J_\nu(z)$ and $J_{\nu+1}(z)$ are Bessel functions of orders ν and $\nu+1$ with the addition that h is a real constant and ν and z are not necessarily real.

The object of the present scheme is to extend the results concerning zeros of (1.1) to that of the function

$$G_\nu(z) = \{f(z)+\nu\}\;J_\nu(z) - z\,J_{\nu+1}(z) \qquad (1\cdot2)$$

where $f(z)$ is analytic throughout in the complex plane.

It is essential to note that the zeros of $G_\nu(z)$ associated with even function $f(z)$ are symmetrically distributed with respect to the origin, whereas the zeros of $G_\nu(z)$ assooiated with odd function $f(z)$ do not obey such law in general.

By virtue of Lommel's result[2] we obtain

$$\int_0^1 t\,J_\nu(\alpha_0 t)\,J_\nu(\alpha t)\,dt = \frac{J_\nu(\alpha_0)\,J_\nu(\alpha)}{\alpha_0^2-\alpha^2}\Big\{f(\alpha_0)-f(\alpha)\Big\} \qquad (1.3)$$

subject to the assumption that α_0 and α are any two distinct zeros of $G_\nu(z)$ and $\nu+1>0$; which evidently furnishes the idea that the common zeros of $f(z)$ and $G_\nu(z)$ are real.

On the other hand, the existence of real zeros of $G_\nu(z)$ under the case $f(\alpha_0) \neq f(\alpha)$ may be proved by introducing inconsistency in (1.3), when α_0 is a complex number and α its complex conjugate.

In particular, if $f(z) = \genfrac{}{}{0pt}{}{\text{Cos}}{\text{Sin}}(z)$ we obtain

$$\int_0^1 t\, J_\nu(\alpha_0 t)\, J_\nu(\alpha t)\, dt = \begin{cases} > \dfrac{J_\nu(\alpha_0)\, J_\nu(\alpha)}{\alpha_0^2 - \alpha^2} \left\{ \operatorname{Cos} \alpha_0 - \operatorname{Cos} \alpha \right\} \\ > \dfrac{J_\nu(\alpha_0)\, J_\nu(\alpha)}{\alpha_0^2 - \alpha^2} \left\{ \operatorname{Sin} \alpha_0 - \operatorname{Sin} \alpha \right\} \end{cases}$$

accordingly.

Again, assuming α_0 a complex number and α its complex conjugate, we get

$$\int_0^1 t\,(P^2+Q^2)\, dt = \begin{cases} > \left\{ \dfrac{R^2+S^2}{\varrho^2 \operatorname{Sin} 2\omega} \right\} \operatorname{Sin} h\,(\varrho \sin \omega) \cos(\varrho \cos \omega) \\ > -\left\{ \dfrac{R^2+S^2}{\varrho^2 \operatorname{Sin} 2\omega} \right\} \operatorname{Sin} h\,(\varrho \sin \omega) \sin(\varrho \cos \omega) \end{cases}$$

where P, Q, R, S are real numbers, proving thereby a contradiction. Therefore the zeros of the functions $\left\{ \begin{matrix} \operatorname{Sin} \\ \operatorname{Cos} \end{matrix} (z) + \nu \right\} J_\nu(z) - z\, J_{\nu+1}(z)$ are real for $\nu + 1 > 0$.

The existence of real zeros of $G_\nu(z)$ associated with the Bessel function $(z/2)^{-\mu} \times J_\mu(z)$ and the Struve function $(z/2)^{-\mu-1} H_\mu(z)$ can not be traced by virtue of the relation (1.3), and thus it becomes essential to establish a process to decide the reality of zeros of $G_\nu(z)$.

In the present paper an attempt has been made to derive certain results concerning zeros of $G_\nu(z)$ associated with even functions in succession and it has been shown that the existence of real zeros depends upon the choice of the function $f(z)$ and the value of the parameter ν unlike the case under (1.1) which furnishes real zeros of $F_\nu^{(h)}(z)$ for all h and ν satisfying $h + \nu > 0$ and two purely imaginary zeros for all values of h and ν satisfying $h + \nu < 0$ (2). It may however be mentioned that the zerosof $G_\nu(z)$ associated with odd function $f(z)$ can not be

determined by the method described later in Sec. 4 due to the unsymmetrical situations of zeros with respect to the origin, but the zeros may be determined by trial and error method which helps to determine the zeros 3.7 and (−4.026) of the function $\sin z \; J_0(z) - z\, J_1(z)$ as calculated by means of Tables 6 and 7.

2. *Number of Zeros of $G_\nu(z)$ in an assigned strip of the complex plane :*

Consider the integral
$$\frac{1}{2\pi i}\int_C \frac{d}{dz}\log\left\{ z^{-\nu}\; G_\nu(z)\right\} dz \qquad (2.1)$$

along the sides of the rectangle with the vertices

$$\pm\, i\left\{ B + \frac{\pi}{2}\, l(\nu)\right\},\; \pm\, i\, B + m\,\pi + \tfrac{1}{4}\,\pi + \tfrac{1}{2}\,\nu\pi$$

for large values of B and large integral values of m. Obviously, the number of zeros of $z^{-\nu}\; G_\nu(z)$ inside the rectangular strip is equal to the integral (2.1), as calculated by Cauchy's theorem on Residues.

By virtue of the asymptotic expansion (3)

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\left[\left\{1+\eta_{1,\nu}(z)\right\}\exp\left\{ i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right\} + \left\{1+\eta_{2,\nu}(z)\right\}\exp\left\{-i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right\}\right]$$

for large values of $|z|$, we find that

$$\frac{d}{dz}\log\left\{ z^{-\nu}\; G_\nu(z)\right\} = \frac{\left\{\nu - f(z)\right\} J_{\nu+1}(z) + \left\{f'(z) - z\right\} J_\nu(z)}{\left\{\nu + f(z)\right\} J_\nu(z) - z\, J_{\nu+1}(z)} \qquad (2.2)$$

$$\sim -\left\{\frac{J_{\nu+1}(z)}{J_\nu(z)}\right\}\times$$

$$\left[\frac{f(z)-\nu+\frac{1}{i}\left\{z-f'(z)\right\}\left\{1-\frac{2\nu+1}{2iz}+0\left(z^{-2}\right)\right\}\left\{1+0\left(e^{2iz}\right)\right\}}{f(z)+\nu-iz\left\{1+\frac{2\nu+1}{2iz}+0\left(z^{-2}\right)\right\}\left\{1+0\left(e^{2iz}\right)\right\}}\right] \quad (2.3)$$

for large values of $|z|$.

2a. *Special cases* :

(i) Replacing $f(z)$ by $\sum_{m=0}^{\infty} A_m z^{2m}$, we get

$$\frac{d}{dz}\log\left\{z^{-\nu}\, G_\nu(z)\right\}$$

$$\sim -\left\{\frac{J_{\nu+1}(z)}{J_\nu(z)}\right\}\left\{\frac{\sum_{m=0}^{l} A_m z^{2m} - \nu}{\sum_{m=0}^{l} A_m z^{2m} + \nu}\right\}\times$$

$$\left[\frac{1+\frac{1}{i}\left\{\frac{z-2\sum_{m=0}^{l-1} A_{m+1} z^{2m+1}}{\sum_{m=0}^{l} A_m z^{2m}+\nu}\right\}\left\{1-\frac{2\nu+1}{2iz}+0\left(z^{-2}\right)\right\}\left\{1+0\left(e^{2iz}\right)\right\}}{1-\left\{\frac{zi}{z+f(z)}\right\}\left\{1+\frac{2\nu+1}{2iz}+0\left(z^{-2}\right)\right\}\left\{1+0\left(e^{2iz}\right)\right\}}\right]$$

$$\approx -\frac{J_{\nu+1}(z)}{J_\nu(z)}$$ for sufficiently large values of $|z|$.

(ii) Assuming $f(z) \sim z^{-\frac{1}{2}}\,\psi(z)\sum_{m=0}^{\infty} A_m z^{-m}$ for large values of

$|z|$, where $\Psi(z)$ is a bounded function, as observed in case of Bessel and Struve functions etc. (4)

We obtain

$$\frac{d}{dz} \log \left\{ z^{-\nu} G_\nu (z) \right\}$$

$$\sim - \left\{ \frac{J_{\nu+1}(z)}{J_\nu (z)} \right\} \times$$

$$\left[\frac{z^{\frac{1}{2}} \cdots \nu + \frac{1}{i} \left(z + \frac{1}{2} z^{-3/2} \right) \left(1 - \frac{2\nu+1}{2iz} + 0 \left(z^{-2} \right) \left\{ 1 + 0 \left(e^{2iz} \right) \right\} \right)}{z^{\frac{1}{2}} + \nu \cdots iz \left(1 + \frac{2\nu+1}{2iz} + 0 \left(z^{-2} \right) \right) \left\{ 1 + 0 \left(e^{2iz} \right) \right\}} \right]$$

$$\approx \frac{J_{\nu+1}(z)}{J_\nu (z)} \quad \text{for sufficiently large values of } |z|.$$

(*iii*) Assuming $f(z) \sim e^{-az} z^p \sum_{m=0}^{\infty} A_m z^{-m}$ $(a > 0)$ for large values of $|z|$, as seen in the case of Whittaker and parabolic cylinder functions * etc. we get.

$$\frac{d}{dz} \log \left\{ z^{-\nu} G_\nu (z) \right\}$$

---

* $W_{k,m}(z) \sim e^{-z/2} \cdot z^k \left\{ 1 + \sum_{n=1}^{\infty} \frac{\left\{ m^2 - (k - \frac{1}{2})^2 \right\} \cdots \left\{ m^2 - (k - n + \frac{1}{2})^2 \right\}}{\lfloor n \; z^n} \right\}$,

for large values of $|z|$ when $|\arg z| \leqslant \pi - \alpha < \pi$

as given by Whittaker and Watson in the Book 'A Course of modern Analysis' Cambridge 2nd Edn. (1958), p. 343.

$$\sim -\left\{\frac{J_{\nu+1}(z)}{J_\nu(z)}\right\} \times$$

$$\left[\frac{e^{-az} z^p -\nu+\frac{1}{i}\left\{z+e^{-az} z^p\left(\frac{p}{z}-a\right)\right\}\left\{1 \frac{2\nu+1}{2iz}+0\left(z^{-2}\right)\right\}}{e^{-az} z^p+\nu -zi\left\{1+\frac{2\nu+1}{2iz}+0\left(z^{-2}\right)\right\}} \qquad \frac{\left\{1+0\left(e^{2iz}\right)\right\}}{\left\{1+0\left(e^{2iz}\right)\right\}}\right]$$

$$\approx -\frac{J_{\nu+1}(z)}{J_\nu(z)} \text{ for sufficiently large values of } |z| .$$

2b. *Conclusion* : Using the result

$$-\frac{1}{2\pi i}\int_C \frac{J_{\nu+1}(z)}{J_\nu(z)} dz = m+0\left(\frac{1}{m}\right) \tag{5}$$

We conclude that the number of zeros of $z^{-\nu} G_\nu(z)$ between the imaginary axis and the line on which

$$R(z) = m\pi+\frac{\pi}{2} R(\nu)+\tfrac{1}{4}\pi$$

is precisely equal to m .

3. $G_\nu(z)$ *as an infinite product.*

We shall begin with the assumption that $G(\pm g_{\nu,i}) = 0$ for all positive integral values of i, such that $g_{\nu,i} \neq g_{\nu,j}$

for all $i \neq j$, with the addition that $\left\{ \begin{matrix} \text{Real part} \\ \text{Imaginary part} \end{matrix} \left(g_{\nu,i} \right) \right\} > 0$

and $\left\{ \begin{matrix} \text{Real part} \\ \text{Imaginary part} \end{matrix} \left(g_{\nu,i} \right) \right\} < \left\{ \begin{matrix} \text{Real part} \\ \text{Imaginary part} \end{matrix} (g_{\nu,j}) \right\}$ for all i,j satisfying $0 < i < j$.

Evaluating the integral

$$\frac{l}{2\pi i} \int_D \frac{z}{\omega(\omega - z)} \frac{d}{d\omega} \left\{ \omega^{-\nu} G(\omega) \right\} d\omega \tag{3.1}$$

taken along the rectangle D with its vertices $\pm A \pm iB$ for large values of A and B, containing the zero $g_{\nu,m}$ ($m \equiv a$ positive integer) of the highest rank of $G_\nu(z)$, with respect to the poles $z, \pm g_{\nu,i}$ ($i \equiv a$ positive integer) we get

$$\frac{l}{2\pi i} \int_D \frac{z}{\omega(\omega - z)} \frac{d}{d\omega} \log \left\{ \omega^{-\nu} G_\nu(\omega) \right\} d\omega$$

$$= \frac{d}{dz} \log \left\{ z^{-\nu} G_\nu(z) \right\} - \sum_{i=}^{m} \left\{ \frac{1}{z - g_{\nu,i}} + \frac{1}{g_{\nu,i}} \right\} - \sum_{i=1}^{m} \left\{ \frac{1}{z + g_{\nu,i}} - \frac{1}{g_{\nu,i}} \right\}$$

subject to the restriction that the integrand is analytic inside and upon the rectangle D.

In other words we obtain the result

$$\frac{d}{dz} \log \left\{ z^{-\nu} G_\nu(z) \right\} = \sum_{i=1}^{m} \left\{ \frac{1}{z - g_{\nu,i}} + \frac{1}{g_{\nu,i}} \right\} + \sum_{i=1}^{m} \left\{ \frac{1}{z + g_{\nu,i}} - \frac{1}{g_{\nu,i}} \right\}$$

Integrating with respect to z between the limits $(0, z)$ we obtain

$$\log \left\{ \frac{2^\nu \Gamma(\nu+1) G_\nu(z)}{z^\nu \{ f(\nu) + \nu \}} \right\} = \sum_{i=1}^{\infty} \log \left\{ \left(1 - \frac{z}{g_{\nu,i}}\right) \exp (z/g_{\nu,i}) \right\} +$$

$$+ \sum_{i=1}^{\infty} \log \left\{ \left(1 + \frac{z}{g_{\nu,i}} \right) \exp \left(- z/g_{\nu,i} \right) \right\}$$

Or, simply we have

$$G_\nu(z) = \left\{ \frac{f(0)+\nu}{\Gamma(\nu+1)} \right\} (z/2)^\nu \prod_{i=1}^{\infty} \left(1 - \frac{z^2}{j^2_{\nu,i}} \right) \tag{3.2}$$

3a. *Verification :* On putting $f(z) = h$ (Real constant), (3.2) reduces to the form

$$G_\nu(z) \rightarrow F_\nu^{(h)}(z) = \left\{ \frac{h+\nu}{\Gamma(\nu+1)} \right\} (z/2)^\nu \prod_{i=1}^{\infty} \left(1 - \frac{z^2}{j^2_{\nu,i}} \right)$$

a result proved previously (1).

3b. *Boundedness of* $\frac{d}{dz} \log \left\{ z^{-\nu} G_\nu(z) \right\}$

Choosing $A = n\pi + \frac{1}{2}\pi R(\nu) + \frac{1}{4}\pi$

and $B = \frac{1}{2}\pi I(\nu) + B_1$, we find

$$\left| \frac{d}{dz} \log \left\{ z^{-\nu} G_\nu(z) \right\} \right| < \frac{K_2 |\nu - f(z)| + |z f'(z)|}{K_1 |z| \, |\nu + f(z)|},$$

where K_1, K_2 are positive numbers satisfying the inequality

$$K_1 < \left| \frac{J_{\nu+1}(z)}{J_\nu(z)} \right| < K_2 . \, *$$

---

* $\frac{J_{\nu+1}(z)}{J_\nu(z)} \sim i \left\{ \frac{1 + \eta_{2,\nu+1}(z)}{1 + \eta_{2,\nu}(z)} \right\} \left\{ 1 + O\left(e^{2iz}\right) \right\}$ for large values of $|z|$,

where $\eta_{2,\nu}(z) = 1 + \frac{4\nu^2 - 1}{8iz} + O(z^{-2})$ (3)

Moreover, assuming $\left|\dfrac{\nu - f(z)}{\nu + f(z)}\right| < K_3$, $\left|\dfrac{z - f'(z)}{\nu - f(z)}\right| < K_4$ *and*

$\left|\dfrac{\nu + f(z)}{z}\right| < K_5$ for finite values of K_3, K_4 and K_5 (which is possible according to 2a (*i*)) we get

$$\left|\frac{d}{dz}\log\left\{ z^{-\nu}\, G_\nu(z)\right\}\right| < \frac{K_3 K_5 (K_2 + K_4)}{K_1 - K_5} = K \quad \text{(say)} \tag{3b.1}$$

Also, introducing the inequalities

$\left|\dfrac{z - f(z)}{z}\right| < L_1$, $\left|1 - \dfrac{f'(z)}{z}\right| < L_2$ and $\left|\dfrac{\nu + f(z)}{z}\right| < L_3$ for

finite values of L_1, L_2, L_3 (according to 2a (*ii*) and (*iii*)) we obtain

$$\left|\frac{d}{dz}\log\left\{ z^{-\nu}\, G_\nu(z)\right\}\right| < \frac{K_2 L_1 + L_2}{K_1 - L_1} = L \quad \text{(say)} \tag{3b.2}$$

which establishes the boundedness of $\dfrac{d}{dz}\log\left\{ z^{-\nu}\, G_\nu(z)\right\}$.

4. *Idea of the existence of real zeros of $G_\nu(z)$ associated with even function.*

Differentiating (3.2) logarithmically we get

$$\frac{d}{dz}\log\left\{ z^{-\nu}\, G_\nu(z)\right\} = -\sum_{n=1}^{\infty} \frac{2z}{g^2_{\nu,n} - z^2} =$$

$$-2\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} z^{2m+1} \Big/ g_{\nu,n}^{2m+2},$$

provided that $|z| < g_{\nu,n}$.

Changing the order of summations, a process easily justifiable, and assuming $\sigma_{\nu,1}^{(m)} = \sum_{i=1}^{\infty} 1/g_{\nu,i}^{2m}$ for positive integral values of m, we arrive at the result

$$\left\{ \nu - f(z) \right\} J_{\nu+1}(z) + \left\{ f'(z) - z \right\} J_{\nu}(z) = 2 \left[z J_{\nu+1}(z) - \left\{ \nu + f(\nu) \right\} J_{\nu}(z) \right] \times \sum_{m=0}^{\infty} z^{2m+1} \sigma_{\nu,1}^{(m+1)} \tag{4.1}$$

Replacing $f(z)$ by a known function (even) it is possible to determine the numbers $\sigma_{\nu,1}^{(m)}$ (m = a positive integer) by comparing the coefficients of odd powers of z on both the sides in (4.1) and the inequalities satisfied by the m^{th} zero $g_{\nu,m}$ and the numbers $\sigma_{\nu,m}^{(r)}$ may readily be obtained in the form

$$\left[\sigma_{\nu,m}^{(r)} \right]^{-1/2r} < g_{\nu,m} < \left[\sigma_{\nu,m}^{(r)} / \sigma_{\nu,m}^{(r+1)} \right]^{\frac{1}{2}}, \text{ (m and r are positive integers)}$$

where $\quad \sigma_{\nu,m}^{(r)} = \sum_{p=m}^{\infty} 1/g_{\nu,p}^{2r}$.

It is remarkable to note that the length of the interval $\left\{ \left[\sigma_{\nu,m}^{(r)} \right]^{-1/2r}, \left[\sigma_{\nu,m}^{(r)} / \sigma_{\nu,m}^{(r+1)} \right]^{\frac{1}{2}} \right\}$ decreases with the increase of the value of r, which ultimately gives rise to a process for calculating the real zeros of $G_{\nu}(z)$.

4a. *Special cases*

(i) *Determination of σ—numbers corresponding to $G_{\nu}(z)$ associated with the polynomial* $\sum_{m=0}^{l} A_m z^{2m}$.

Replacing $f(z)$ by $\sum\limits_{m=0}^{l} A_m z^{2m}$, (4.1) assumes the form

$$\left\{ \nu - \sum_{m=0}^{l} A_m z^{2m} \right\} J_{\nu+1}(z) + \sum_{m=0}^{l-1} \left\{ 2(m+1) A_{m+1} z^{2m+1} - z \right\} J_\nu(z)$$

$$=2 \left[z J_{\nu+1}(z) - \left\{ \nu + \sum_{m=0}^{l} A_m z^{2m} \right\} J_\nu(z) \right] \sum_{m=0}^{\infty} z^{2m+1} \sigma_{\nu, A_i, 1}^{(m+1)} \quad (4a.1)$$

where $\sigma_{\nu, Ai, 1}^{(m)} = \sum\limits_{r=1}^{\infty} 1/ g_{\nu, Ai, r}^{2m}$ ($i \equiv$ a positive integer)

Comparing the coefficients of $z^{\nu+1+2p}$ $(p=0, 1, 2, \ldots)$ we obtain

$$\sigma_{\nu, A_i, 1}^{(1)} = \frac{A_0+\nu+2-4A_1(\nu+1)}{4(A_0+\nu)(\nu+1)} \quad (4a.2)$$

and $$\sum_{n=1}^{l} \sum_{m=1}^{p+1-n} \frac{(-)^{n+m+1} 4^{n+m} A_n \sigma_{\nu, A_i, 1}^{(m)}}{\Gamma(p+2-m-n)\, \Gamma(\nu+p+2-m-n)} +$$

$$\sum_{m=1}^{p+1} \left[(-)^{m+1} 4^m \left\{ \nu + A_0 + 2(p-m+1) \right\} \times \frac{\sigma_{\nu, A_i, 1}^{(m)}}{\Gamma(p+2-m)\, \Gamma(\nu+p+2-m)} \right]$$

$$=(p+1) \sum_{m=1}^{l} \frac{(-)^m 4^m A_m}{\Gamma(p+2-m)\, \Gamma(\nu+p+2-m)} + \frac{A_0+\nu+2(p+1)}{\Gamma(p+1)\, \Gamma(\nu+p+2)} \quad (4a.3)$$

4b. *Relation involving the numbers* $\sigma_{\nu, A_i, 1}^{(m)}$ and $\sigma_{\nu, A_0, 1}^{(m)}$

Combining the result

$$\sum_{m=1}^{p+1} \frac{(-)^{m+1}\,4^m\,\{\nu+A_o+2\,(p-m+1)\}}{\Gamma(p+2-m)\,\Gamma(\nu+p+2-m)}\,\sigma^{(m)}_{\nu,A_o,1} = \frac{A_o+\nu+2\,(p+1)}{(p+1)\,\Gamma(\nu+p+2)} \quad (1)$$

with (4a.2) and (4a.3) we get

$$\sigma^{(1)}_{\nu,A_l,1} = \sigma^{(1)}_{\nu,A_o,1} - \frac{A_1}{A_o+\nu} \tag{4b.1}$$

and $\sum_{m=1}^{p+1} (-4)^m\,(-1-p)_m\,(-\nu-p-1)_m \left\{\nu+A_o+2\,(p-m+1)\right\}$

$$\left\{\sigma^{(m)}_{\nu,A_o,1} - \sigma^{(m)}_{\nu,A_l,1}\right\}$$

$$=(p+1)\sum_{m=1}^{l} (-4)^m\,(-1-p)_m\,(-\nu-p-1)_m\,A_m +$$

$$\sum_{n=1}^{l}\ \sum_{m=1}^{p+1-n}\left[(-4)^{m+n}(-1-p)_{m+n}(-\nu-p-1)_{m+n}\times A_n\ \sigma^{(m)}_{\nu,A_l,1}\right]^* \tag{4b,2}$$

4c. *Verification* : On putting $A_i=0$ for $i=1, 2, \ldots l$, (4b. 1) and (4b.2) reduce to $\sigma^{(1)}_{\nu,A_l,1} \rightarrow \sigma^{(1)}_{\nu,A_o,1}$

and $\sum_{m=1}^{p+1} (-4)^m\,(-1-p)_m\,(-\nu-1-p)_m \left\{\nu+A_o+2\,(p-m+1)\right\}$

$$\left\{\sigma^{(m)}_{\nu,A_l,1} - \sigma^{(m)}_{\nu,A_o,1}\right\}=0,$$

consistent with all positive integral values of p.

Hence $\sigma^{(m)}_{\nu,A_l,1} \rightarrow \sigma^{(m)}_{\nu,A_o,1}$ for $A_i=0$ ($i=1, 2,.. l$), proving the validity of the result (4a.1).

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*$(\alpha)_m$ stands for $\frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}$ as used by Whittaker and Watson in the book 'A Course of Modern Analysis' Cambridge, Fourth Edition, 1962

4d. *Relation involving* $\sigma^{(m)}_{3/2,\ A_l,1}$ *and Bernoullian numbers* : By virtue of the recurrence relation $\frac{2\nu}{z} J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z)$

(4a.1) may be cast into the form

$$\left\{ 2\nu\left(\nu - \sum_{m=0}^{l} A_m z^{2m} \right) + \sum_{m=0}^{l-1} 2(m+1) A_{m+1} z^{2m+2} - z^2 \right\} J_\nu(z) -$$

$$\left\{ \nu - \sum_{m=0}^{l} A_m z^{2m} \right\} z J_{\nu-1}(z)$$

$$= 2\left[\left\{ \nu - \sum_{m=0}^{l} A_m z^{2m} \right\} z J_\nu(z) - z^2 J_{\nu-1}(z) \right] \sum_{m=0}^{\infty} z^{2m+1} \sigma^{(m+1)}_{\nu,\ A_l,1} \,.$$

On putting $\nu = 3/2$, the relation further reduces to the form

$$\left\{ 3\left(\frac{3}{2} - \sum_{m=0}^{l} A_m z^{2m} \right) + \sum_{m=0}^{l-1} 2(m+1) A_{m+1} z^{2m+2} - z^2 \right\}$$

$$\left\{ 1 - z \cot z \right\} - z^2 \left(\frac{3}{2} - \sum_{m=0}^{l} A_m {}^{2m} \right)$$

$$= 2\left\{ z\left(\frac{3}{2} - \sum_{m=0}^{l} A_m z^{2m} \right)\left(1 - z \cot z \right) - z^3 \right\} \sum_{m=0}^{\infty} z^{2m+1} \sigma^{(m+1)}_{\frac{3}{2},\ A_l,1}$$

in accordance with the result $J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \operatorname{Sin} z$.

In other words, we have

$$\left\{ 3\left(\frac{3}{2} - \sum_{m=0}^{l} A_m z^{2m} \right) + \sum_{m=0}^{l-1} 2(m+1) A_{m+1} z^{2m+2} - z^2 \right\} \sum_{m=1}^{\infty} \frac{(2z)^{2m} B_m}{\lfloor 2m}$$

$$- z^2 \left(\frac{3}{2} - \sum_{m=0}^{l} A_m z^{2m} \right)$$

$$= 2\left\{ z\left(\frac{3}{2} - \sum_{m=0}^{l} A_m\, z^{2m}\right) \sum_{m=1}^{\infty} \frac{(2z)^{2m} B_m}{\lfloor 2m} \, z^{-1} \right\} \sum_{m=0}^{\infty} z^{2m+1}\, \sigma^{(m+1)}_{\frac{3}{2},\, A_i, 1}$$

where B_m is the m^{th} Bernoullian number.

Comparing the coefficients of z^{2r+2} we get

$$2\left\{ \sum_{q=1}^{l} \sum_{p=1}^{r+1-2q} \frac{A_q\, 2^{2(p+q-1)}\, B_{p+q-1}}{\lfloor 2p+2q-2} \, \sigma^{(r+2-p-2q)}_{\frac{3}{2},\, A_i, 1} \right.$$

$$\left. - \left(\frac{3}{2} - A_0\right) \sum_{p=1}^{r} \frac{2^{2p} B_p}{\lfloor 2p} \, \sigma^{(r+1-p)}_{\frac{3}{2},\, A_i, 1} + \sigma^{(r)}_{\frac{3}{2},\, A_i, 1} \right\}$$

$$= 3 \sum_{p=1}^{l} \frac{A_p\, 2^{2(r+1-p)}}{\lfloor 2(r+1-p)} B_{r+1-p} - 3\left(\frac{3}{2} - A_0\right) \frac{2^{2r+2} B_{r+1}}{\lfloor 2r+2} + \frac{2^{2r} B_r}{\lfloor 2r}$$

$$- 2 \sum_{p=1}^{l} \frac{p\, A_p\, 2^{2(r-p+1)}}{\lfloor 2(r+1-p)} B_{r+1-p} - A_r, \qquad (4d{\cdot}1)$$

where r is a positive integer.

4e. **Verification** : On putting $A_i = 0$ for $i = 1, 2, \ldots l$,(4d·1)

reduces to the form

$$2\,\sigma^{(r)}_{\frac{3}{2},\, A_0, 1} = \left(\frac{3}{2} - A_0\right)\left[\, 2 \sum_{p=1}^{r} \frac{2^{2p}\, B_p}{\lfloor 2p} \, \sigma^{(r+1-p)}_{\frac{3}{2},\, A_0, 1} - \frac{3.2^{2r+2}}{\lfloor 2r} B_{r+1} \right]$$

$$+ \frac{2^{2r} B_r}{\lfloor 2r},$$

a result derived previously (1) .

4f. *Determination of zeros of* G_ν (z) *associated with the series* $\sum_{m=0}^{\infty} A_m z^{2m}$:

Taking the limit l tending to infinity into consideration, (4a.3) assumes the form

$$\sum_{n=1}^{\infty} \sum_{m=1}^{p+1-n} \frac{(-)^{n+m+1} 4^{n+m} A_n \sigma^{(m)}_{\nu, A_i, 1}}{\Gamma(p+2-m-n)\, \Gamma(\nu+p+2-m-n)} +$$

$$\sum_{m=1}^{p+1} \frac{(-)^{m+1} 4^m \{\nu+A_o+2(p-m+1)\} \sigma^{(m)}_{\nu, A_i, 1}}{\Gamma(p+2-m)\, \Gamma(\nu+p+2-m)}$$

$$= (p+1) \sum_{m=1}^{\infty} \frac{(-4)^m A_m}{\Gamma(p+2-m)\, \Gamma(\nu+p+2-m)} + \frac{A_o+\nu+2(p+1)}{\Gamma(p+1)\, \Gamma(\nu+p+2)} \tag{4f.1}$$

together with result (4a.2); provided that the series

$\sum_{m=0}^{\infty} A_m z^{2m}$, $\sum_{m=1}^{\infty} \frac{(-4)^m A_m}{\Gamma(p+2-m)\, \Gamma(\nu+p+2-m)}$ and

$\sum_{n=1}^{\infty} \sum_{m=1}^{p+1-n} \frac{(-)^{n+m+1} 4^{n+m} A_n \sigma^{(m)}_{\nu, A_i, 1}}{\Gamma(p+2-m-n)\, \Gamma(\nu+p+2-m-n)}$ converge for all positive integral values of p.

Special cases :

(i) *Real Zeros of the function* $J_1(x) \{2 J_o(z) - z^2\}$

Replacing $f(z)$ by $\frac{2}{z} J_1(z)$ we obtain

$$G_o(z) \equiv J_1(z) \left\{ \frac{2}{z} J_o(z) - z \right\} \text{ by virtue of } \tag{1.2}$$

Obviously, there are infinite number of real zeros of $G_0(z)$ common to those of $J_1(z)$ which has already been given by Watson [6]. The zeros corres

ponding to the function $\left\{ \frac{2}{z} J_0(z) - z \right\}$ may be calculated by means of (4a.2) and (4a.3), as incorporated in the form 4

Table 1

r	$\left[\sigma^{(r)}_{0, A_i, 1} \right]^{-1/2r}$	$\left[\sigma^{(r)}_{0, A_i, 1} / \sigma^{(r+1)}_{0, A_i, 1} \right]^{\frac{1}{2}}$
1	1.07	1.277
2	1 16[illegible]	1.175
3	1.171	...

Thus, the smallest zero of G_0 (z) is 1.173 which satisfies the equation $J_0(z) = \frac{1}{2} z^2$, as may be checked by the Table I.(7) The other higher zeros of G_0 (z) may be calculated by virtue of the inequalities

$$\left[\sigma^{(r)}_{\nu,m} \right]^{-1/2r} < z_{\nu,m} < \left[\sigma^{(r)}_{\nu, m} \Big/ \sigma^{(r+1)}_{\nu,m} \right]^{\frac{1}{2}}$$

where $\sigma^{(r)}_{\nu,2} = \sigma^{(r)}_{\nu,1} + \dfrac{1}{(1\cdot173)^{2r}}$

(ii) *Real zeros of the function* $\left\{ \frac{2}{z} J_1(z) + 1 \right\} J_1(z) - z J_2(z)$

Replacing f (z) by $\frac{2}{z}$ $J_1(z)$ we obtain

$$G_2(z) = \left\{ \frac{2}{z} J_1(z) + 1 \right\} J_1(z) - z J_2(z) \quad \text{by viritue of } (1.2).$$

The smallest zero of G_1 (z) may be calculated in the form

Table 2

r	$\left[\sigma^{(r)}_{1, A_i, 1}\right]^{-\frac{1}{2}r}$	$\left[\sigma^{(r)}_{1, A_f, 1} \Big/ \sigma^{(r+1)}_{1, A_i, 1}\right]^{\frac{1}{2}}$	Smallest zero of $G_1(z)$
1	1.789	2.618	$g_{1,1} = 2.175$
2	2.163	2.188	
3	2.172	...	

(iii) *Smallest Zeros of* $G_1(z) \equiv \left\{\frac{2}{z} J_1(z) + 2\right\} J_2(z) - z J_3(z)$

and $G_2{}'(z) \equiv \left\{\frac{4}{z^2} J_2(z) + 2\right\} J_2(z) - z J_3(z)$ *are not real*

We shall now put forward the numerical analysis for tracing the existence of smallest zeros of $G_1(z)$ and $G_2(z)$ successively in the forms

Table 3

r	$\left[\sigma^{(r)}_{2, A_i, 1}\right]^{-\frac{1}{2}r}$	$\left[\sigma^{(r)}_{2, A_i, 1} \Big/ \sigma^{(r+1)}_{2, A_f, 1}\right]^{\frac{1}{2}}$	Smallest zero of $G_2(z)$
1	2.353	4.243	$g_{2,1}$ does not exist on the real axis
2	3.16	3.117	
3	3.148	...	

Table 4

r	$\left[\sigma^{(r)}_{2,\,A_i,\,1}\right]^{-1/2r}$	$\left[\sigma^{(r)}_{2,\,A_i,\,1} / \sigma^{(r+1)}_{2,\,A_i,\,1}\right]^{1/2}$	Smallest zero of $G_2'(z)$
1	2.449	4	
2	3.130	2.249	$g'_{2,1}$ does not exist on the real axis
3	2.804	...	

IV. *Real Zeros of* $G_3(z) = \{\text{Cos}\, z + 1\}\, J_1(z) - z\, J_2(z)$

In order to justify the existence of real zeros of $\{\text{Cos}\, z + 1\}\, J_1(z) - z\, J_2(z)$ (discussed previously) we shall give the numerical analysis in the form

Table 5

r	$\left[\sigma^{(r)}_{1,\,A_i,\,1}\right]^{-1/2r}$	$\left[\sigma^{(r)}_{1,\,A_i,1} / \sigma^{(r+1)}_{1,\,A_i,1}\right]^{\frac{1}{2}}$	Smallest zero of $G_3(z)$
1	1.414	2.089	
2	1.719	1.7645	$g_3 = 1.762$
3	1.761	...	

V. *Positive and Negative zeros of* $G_4(z) = \{\text{Sin}\, z + 1\}\, J_1(z) - z\, J_2(z)$. Apart from the method described in 4, we shall use the trial and error method to justify the unsymmetrical situations of zeros of $G_4(z)$ with respect to the origin.

Obviously by virtue of Table I[8], we obtain

Table 6

z	$J_0(z)$	$J_1(z)$	$G_4(z)$	Positive zero of $G_4(z)$
3	$-.2601$	.3391	$-$ve	
3.5	$-.3801$	.1375	$-$ve	$g_4 = 3.7$
3.7	$-.3992$	.0538	.003	

Table 7

z	$J_0(z)$	$J_1(z)$	$\sin z\, J_0(z) + z J_1(z)$	Negative zero of $G_4(z)$
2.5	$-.0484$	.4971	$+$ve	
3	$-.2601$	.3391	$+$ve	
4	$-.3971$	$-.0660$	0.036	
4.025	$-.3954$	$-.0755$	0.0018	$g_4 = -4.026$
4.027	$-.3952$	$-.0763$	-0.0013	
4.026	$-.3953$	$-.0759$	$+0.0002$	

VI. *Real Zeros of the function* $\Psi(z) \equiv \frac{2}{z} H_0(z)\, J_0(z) - z\, J_1(z)$

$H_0(z)$ is the Struve function of order zero, defined by the series

$$\sum_{r=0}^{\infty} \frac{(-)^r (z/2)^{1+2r}}{\left\{\Gamma\left(r+\frac{3}{2}\right)\right\}^2}, \text{ so that}$$

$$A_0 = \frac{4}{\pi},\ A_1 = -\frac{4}{9\pi},\ A_2 = \frac{4}{225\pi},\ A_3 = \frac{-4}{(105)^2\pi}, \ldots\ldots\ \text{etc.},$$

and hence the smallest zero of $\Psi(z)$ may be calculated in the form

r	$\left[\sigma^{(r)}_{0,A_i,1}\right]^{-1/2r}$	$\left[\sigma^{(r)}_{0,A_i,1} / \sigma^{(r+1)}_{0,A_i,1}\right]^{1/2}$	Smallest zero of $\Psi(z)$
1	1·151	1·424	
2	1·28	1·288	$z = 1·28$
3	1·28	...	

VII. *Real Zeros of the function* $\psi(z) = \{4/z^2 H_1(z) + 1\} J_1(z) - z J_2(z)$, $H_1(z)$ is the Struve function of order unity, defined by the

series $\sum_{r=o}^{8} \frac{(-)^r (z/2)^{2+2r}}{\Gamma(r+3/2)\,\Gamma(r+5/2)}$, so that $A_0 = \frac{8}{3\pi}$, $A_1 = -\frac{8}{45\pi}$,

$A_2 = \frac{8}{1575\pi}$, $A_3 = \frac{-8}{9\pi\,(105)^2}$, ...etc.

and hence the smallest zero of $\psi(z)$ may be calculated in the form

r	$\left[\sigma^{(r)}_{1,A_i,} \right]^{-1/2r}$	$\left[\sigma^{(r)}_{1,A_i,1} / \sigma^{(r+1)}_{1,A_i,1} \right]^{1/2}$	Smallest zero of $\psi(z)$
1	1·854	2·639	
2	2·212	2·255	$z = 2{\cdot}23$
3	2·227	...	

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SECOND ORDER PERTURBATIONS IN POLAR CO-ORDINATES OF AN ARTIFICIAL SATELLITE IN THE GRAVITATIONAL FIELD OF AN OBLATE SPHEROID

By

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ABSTRACT

In this paper the motion of an artificial satellite in the gravitational field of an axially symmetric oblate spheroid has been studied in Dleaunay variables. During this study perturbation up to the second order is taken into account and von Zeipel's method has been exploited. Lastly the radius vector r and the longitude u for the artificial satellite have also been calcutated by the same method and with perturbation stated above.

1. INTRODUCTION

This paper has been written with two fold aims. Firstly, we aim to calculate the second order perturbations in the Delaunay variables in the gravitational field of an axially symmetric oblate spheroid. Since the launching of the first artificial satallite, several papers dealing with the first order perturbations have been published, With the help of the first order theory, information on upper atmosphere and geopotential has been obtained from observations of satellites. It is hoped that with the help of the second order perturbation more accurate information may be obtained.

In articles 1—3, we have studied the second order perturbation in Delaunay variables. We have exploited von Zeipel's method [1] to attain our goal. This method was first used by D. Brouwer to study the first order perturbation in a similar gravitational field of force. Our work does not differ much from that of Yoshide Kozai so far as second order perturbation is concerned. It was due to his paper that we did not try to calculate the long-period and secular perturbations. Yoshide Kozai too calculates the perturbation in radius vector r and the longitude u. He has followed the conventional method which gives very unwieldy expressions.

The main interest of the paper is the expression of the second order perturbation in radius vector r and the longitude u. Different from the conventional methods, we have used von Zeipel's method. This method shows that the determination of S_2 not only gives the second order perturbation in Delaunay variables but in polar coordinates like r and u too.

This paper gives an important result by showing that von Zeipel's method is one of the most powerful methods in the artificial satellite theory. A numerical study of the method will give a comparative aspect of von Zeipel's method with those of others.

It is hoped that the second order theory will help us much in our study of an artificial satellite in some other gravitational field. It was this study which made us to calculate the second order perturbation in our own way to suit our further study.

2. EQUATIONS OF MOTION

The equations of motion of a small mass attracted by an oblate spheroid can be written as [1, page 564]

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l}, & \frac{dl}{dt} &= -\frac{\partial F}{\partial L} \\ \frac{dG}{dt} &= \frac{\partial F}{\partial g}, & \frac{dg}{dt} &= -\frac{\partial F}{\partial G} \\ \frac{dH}{dt} &= \frac{\partial F}{\partial h} & \frac{dh}{dt} &= -\frac{\partial F}{\partial H} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \qquad (1)$$

$$\text{where } F = \frac{\mu^2}{2L^2} + \frac{\mu^4 k_2}{L^6}\left[\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G^2}\right)\left(\frac{L^3}{G^3}+\sigma_1\right)+\left(\frac{3}{2}-\frac{3}{2}\frac{H^2}{G^2}\right)\sigma_2\right]+$$

$$+\frac{\mu^6 k_4}{L^{10}}\left[\left(\frac{3}{8}-\frac{15}{4}\frac{H^2}{G^2}+\frac{35}{8}\frac{H^4}{G^4}\right)\left(\frac{5}{2}\frac{L^7}{G^7}-\frac{3}{2}\frac{L^5}{G^5}+\sigma_3\right)+\right.$$

$$+\left(\frac{-5}{6}+\frac{20}{3}\frac{H^2}{G^2}-\frac{35}{6}\frac{H^4}{G^4}\right)\left\{\left(\frac{3}{4}\frac{L^7}{G^7}-\frac{3}{4}\frac{L^5}{G^5}\right)\cos 2g+\sigma_4\right\}+$$

$$\left.+\left(\frac{35}{24}-\frac{35}{12}\frac{H^2}{G^2}+\frac{35}{24}\frac{H^4}{G^4}\right)\sigma_5\right] \ldots \quad \ldots \quad \ldots \qquad (2)$$

Here it may be noticed that the disturbing function has been considered upto the second order terms different from the disturbing function considered in [1, page 564] where

$$\sigma_1 = \sum_{j=1}^{\infty} 2\,P_j \cos jl, \quad \sigma_2 = \sum_{j=-\infty}^{+\infty} Q_j \cos(2g+jl), \quad \sigma_3 = \sum_{j=1}^{\infty} 2\,P_j' \cos jl$$

$$\sigma_4 = \sum_{j=1}^{+\infty} 2\,P_j'' \cos(2g+jl), \quad \sigma_5 = \sum_{j=-\infty}^{+\infty} Q_j'' \cos(4g+jl)$$

3. SOLUTION OF THE PROBLEM

From the expression of F given in (2) it is clear that h is an ignorable coordinate and so H is constant. Now for the complete solution we shall consider a contact transformation from Delaunay variables L, G, H, l, g, h to new variables L', G', H', l', g' and h' with the aid of the determining function S (L', G', H', l, g, h) where

$$\left.\begin{array}{ll} L = \dfrac{\partial S}{\partial l} & l' = \dfrac{\partial S}{\partial L'} \\[2ex] G = \dfrac{\partial S}{\partial g} & g' = \dfrac{\partial S}{\partial G'} \\[2ex] H = \dfrac{\partial S}{\partial h} & h' = \dfrac{\partial S}{\partial H'} \end{array}\right| \quad \ldots \quad \ldots \quad \ldots (3)$$

The determining function S will be obtained by von Zeipel's method (1916) such that l' is absent from our Hamiltonian F*. This is known as elimination of short-period terms from F. Let us choose our determining function S expanded in according to the orders of k_2, *i. e.,* $S = S_0 + S_1 + S_2 + \ldots\ldots\ldots$

where $S_0 = L'l + G'g + H'h$ and let the new Hamiltonian F* be also expanded in terms of the same orders of k_2 & is given by

$$F^* = F_0^* + F_1^* + F_2^* + \ldots\ldots\ldots$$

Applying this transformation, we get

$$F(L, G, H, l, g, -) = F^*(L', G', H', -, g', -)$$

Substituting for L,G,H,l',g',h' from (3) and equating the different orders of terms in k_2, on the two sides, we get

Order zero, $F_0(L') = F_0^*(L') \ldots \quad \ldots \quad \ldots \quad (4)$

Order 1 in k_2 : $\dfrac{\partial F_0}{\partial L'} \dfrac{\partial S'}{\partial l} + F_1 = F_1^* \ldots \quad \ldots \quad \ldots (5)$

Order 2 in k_2 : $\dfrac{\partial F_0}{\partial L'} \dfrac{\partial S_2}{\partial l} + \frac{1}{2} \dfrac{\partial^2 F_0}{\partial L'^2}\left(\dfrac{\partial S_1}{\partial l}\right)^2 + \dfrac{\partial F_1}{\partial l'} \dfrac{\partial S_1}{\partial l} + \dfrac{\partial F_1}{\partial G'} \dfrac{\partial S_1}{\partial g} +$

$$F_2(L', G', H', l', g', -) = F_2^*(L', G', H', -, g', -) \quad \ldots \quad (6)$$

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$$\text{where } F = \frac{\mu^2}{2L^2} + \frac{\mu^4 k_2}{L^6}\Big[\Big(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G^2}\Big)\Big(\frac{L^3}{G^3}+\sigma_1\Big)+\Big(\frac{3}{2}-\frac{3}{2}\frac{H^2}{G^2}\Big)\sigma_2\Big] +$$

$$+\frac{\mu^6 k_4}{L^{10}}\Big[\Big(\frac{3}{8}-\frac{15}{4}\frac{H^2}{G^2}+\frac{35}{8}\frac{H^4}{G^4}\Big)\Big(\frac{5}{2}\frac{L^7}{G^7}-\frac{3}{2}\frac{L^5}{G^5}+\sigma_3\Big)+$$

$$+\Big(\frac{-5}{6}+\frac{20}{3}\frac{H^2}{G^2}-\frac{35}{6}\frac{H^4}{G^4}\Big)\Big\{\Big(\frac{3}{4}\frac{L^7}{G^7}-\frac{3}{4}\frac{L^5}{G^5}\Big)\cos 2g+\sigma_4\Big\}+$$

$$+\Big(\frac{35}{24}-\frac{35}{12}\frac{H^2}{G^2}+\frac{35}{24}\frac{H^4}{G^4}\Big)\sigma_5\Big] \quad \dots \quad \dots \quad \dots \qquad (2)$$

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The determining function S will be obtained by von Zeipel's method (1916) such that l' is absent from our Hamiltonian F^*. This is known as elimination of short-period terms from F. Let us choose our determining function S expanded in according to the orders of k_2, *i. e.*, $S = S_0 + S_1 + S_2 + \ldots\ldots\ldots$

where $S_0 = L'l + G'g + H'h$ and let the new Hamiltonian F^* be also expanded in terms of the same orders of k_2 & is given by

$$F^* = F_0^* + F_1^* + F_2^* + \ldots\ldots\ldots$$

Applying this transformation, we get

$$F(L, G, H, l, g, -) = F^*(L', G', H', -, g', -)$$

Substituting for L, G, H, l', g', h' from (3) and equating the different orders of terms in k_2, on the two sides, we get

Order zero, $F_0(L') = F_0^*(L')\ldots$ $\qquad \cdots \qquad \cdots \qquad (4)$

Order 1 in k_2 : $\dfrac{\partial F_0}{\partial L'}\dfrac{\partial S'}{\partial l} + F_1 = F_1^*\ldots$ $\qquad \cdots \qquad \cdots (5)$

Order 2 in k_2 : $$\frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 + \frac{\partial F_1}{\partial l'}\frac{\partial S_1}{\partial l} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g} + F_2(L', G', H', l', g', -) = F_2^*(L', G', H', -, g', -) \quad \ldots \quad (6)$$

The first order determining function S_1 has been already obtained by Brouwer (1959) as

$$S_1 = \frac{\mu^2 k_2}{G'^3}\left\{ A\,(f - l + e\sin f) + B\,.\,\tfrac{1}{2}\sin(2g+2f) + \frac{e}{2}\sin(2g+f) + \right.$$
$$\left. + \frac{e}{6}\operatorname{Sin}(2g+3f) \right] \quad -(7)$$

$$\text{Where } A = -\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\,,\quad B = \frac{3}{2} - \frac{3}{2}\frac{H^2}{G^2}$$

Then S_2 is derived from the equation (6) in the same manner as S_1 is derived and it is found as :

$$S_2 = \frac{\mu^4 k_2^2 L'^2}{1024\,G'^{6}}\Big[96\left\{-5\eta'^2(1-2\theta'^2-7\theta'^4)+\eta'^4(5-18\theta'^2+5\theta'^4)\right\}(f-l)+$$

$$+\frac{48}{e}\Big[9(11-30\theta'^2+27\theta'^4)-8\eta'^2(17-38\theta'^2+11\theta'^4)-4\eta'^3(1-3\theta'^2)^2(3+\eta'^2)+$$

$$+\eta'^4(53-130\theta'^2-11\theta'^4)\Big]\sin f+6\,[87\,(11-30\theta'^2+27\theta'^4)-$$

$$-2\eta'^2(389-834\theta'^2+437\theta'^4)-64\eta'^3(1-3\theta'^2)+\eta'^4(93-242\theta'^2+77\theta'^4)]\times$$

$$\operatorname{Sin}2f+8e[41(11-30\theta'^2+27\theta'^4)-\eta'^2(191-390\theta'^2+207\theta'^4)-8\eta'^3(1-3\theta'^2)^2]\times$$

$$\operatorname{Sin}3f+24e^2[5(11-30\theta'^2+27\theta'^4)-8\eta'^3(1-3\theta'^2)^2]\times\operatorname{Sin}4f+24e^3\times$$

$$(11-30\theta'^2+27\theta'^4)\operatorname{Sin}5f+2e^4(11-30\theta'^2+27\theta'^4)\times\operatorname{Sin}6f+(1-\theta'^2)\{192e^2\eta'^2\times$$

$$(1-15\theta'^2)(f-l)\cos 2g-12e^4(1-3\theta'^2)\times\operatorname{Sin}(4f-2g)-144e^3(1-3\theta'^2)$$

$$\operatorname{Sin}(3f-2g)+48e^2(1-3\theta'^2)\times(-15+2\eta'^2)\operatorname{Sin}(2f-2g)+48e(1-3\theta'^2)\times$$

$$(-41+15\eta'^2+2\eta'^3)\operatorname{Sin}(f-2g)+\frac{96}{e}\Big[(1-3\theta'^2)(27-9\eta'^3+5\eta'^4)-\eta'^2\times$$

$$(25-3\theta'^2)+2\eta'^4(1+13\theta'^2)\Big]\times\operatorname{Sin}(2g+f)+384[-\eta'^2(1+7\theta'^2)+4\eta'^4\theta'^2]\times$$

$$\operatorname{Sin}(2g+2f)+\frac{32}{e}[(1-3\theta'^2)(-81+27\eta'^3+\eta'^4)+5\eta'^2(11-41\theta'^2)-2\eta'^4\times$$

$$(1-23\theta'^2)]\times\operatorname{Sin}(2g+3f)+12[3(1-3\theta'^2)(-87+16\eta'^3)+2\eta'^2(51-161\theta'^2)-$$

$\eta'^4(1-19\theta'^2)]\ \mathrm{Sin}\ (2g+4f)+48e\ (1-3\theta'^2)\ (-41+7\eta'^2+2\eta'^3)\times$

$\mathrm{Sin}\ (2g+5f)+16e^2(1-3\theta'^2)\ (-45+2\eta'^2)\ \mathrm{Sin}\ (2g+6f)+144e^3\ (3\theta'^2-1)\times$

$\mathrm{Sin}\ (2g+7f)+12e^4\ (3\theta'^2-1)\ \mathrm{Sin}\ (2g+8f)+108e^3\times(1-\theta'^2)\ \mathrm{Sin}\ (f-4g)+$

$9e^4\ (1-\theta'^2)\ \mathrm{Sin}\ \ (2f-4g)+36e\ (1-\theta'^2)\ (-41+29\eta'^2)\ \times\ \mathrm{Sin}\ (4g+f)+$

$3\ [-783\ (1-\theta'^2)+2\eta'^2\ (335-239\theta'^2)+\eta'^4\ (65-257\theta'^2)]\times\mathrm{Sin}\ (4g+2f)+$

$\frac{24}{e}[-81\ (1-\theta'^2)+16\eta'^2\ (4-\theta'^2)+\eta'^4\ (17-65\theta'^2)]\ \mathrm{Sin}\ (4g+3f)+24\times$

$[-\eta'^2\ (51-91\theta'^2)+\eta'^4\ (11-27\theta'^2)]\ \mathrm{Sin}\ (4g+4f)+\frac{24}{e}[81\ (1-\theta'^2)-8\eta'^2\times$

$(15-17\theta'^2)+\eta'^4(39-55\theta'^2)]\mathrm{Sin}(4g+5f)+[2349(1-\theta'^2)-2\eta'^2(861-893\theta'^2)+$

$+\eta'^4\ (157-221\theta'^2)]\ \mathrm{Sin}\ (4g+6f)+36e(1-\theta'^2)\times(41-13\eta'^2)\ \mathrm{Sin}\ (4g+7f)+$

$12e^2\ (1-\theta'^2)\ (45-4\eta'^2)\ \mathrm{Sin}\ (4g+8f)+108e^3\ (1-\theta'^2)\ \mathrm{Sin}\ (4g+9f)+$

$9e^4\ (1-\theta'^2)\ \mathrm{Sin}\ (4g+10f)\ \}]+\frac{\mu^4 k_4}{96G'^7}\ [2(3-30\theta'^2+35\theta'^4)\{6\ \ (f-l)(2+3e^2)+$

$+9(4e+e^3)\mathrm{Sin}\,f+9e^2\ \mathrm{Sin}\ 2f+e^3\mathrm{Sin}\ 3f\}-2(1-8\theta'^2+7\theta'^4)\{60(f-l)e^2\cos 2g+$

$+30(4e+e^3)\ \mathrm{Sin}(2g+f)+20(2+3e^2)\mathrm{Sin}(2g+2f)+10(4e+e^3)\times\mathrm{Sin}(2g+3f)$

$+15e^2\mathrm{Sin}(2g+4f)+2e^3\mathrm{Sin}(2g+5f)+10e^3\mathrm{Sin}(f-2g)\}+(1-2\theta'^2+\theta'^4)\times$

$\{35e^3\mathrm{Sin}(4g+f)+105e^2\ \mathrm{Sin}(4g+2f)+35(4e+e^3)\ \mathrm{Sin}(4g+3f)+35(2+3e^2)\times$

$\mathrm{Sin}(4g+4f)+21(4e+e^3)\mathrm{Sin}(4g+5f)+35e^2\mathrm{Sin}(4g+6f)+5e^3\mathrm{Sin}\ (4g+7f)\}]$

where $\theta'=\frac{H'}{G'}$ and $\eta'=\frac{G'}{L'}$ ... (8)

5. Calculation of Delaunay Variables with Second Order Perturbations.

Taking into consideration of the second order perturbation the six Delaunay variables L, G, H, l, g, h may be given as follows :

$$L=L'+\frac{\partial S_1}{\partial l}+\frac{\partial S_2}{\partial l}$$

$$=L'+\frac{\mu^2 k_2}{L'^3}\Big[\Big(-\frac{1}{2}+\frac{3}{2}\theta'^2\Big)\Big(\frac{a^3}{r^3}-\frac{L'^3}{G'^3}\Big)+\Big(\frac{3}{2}-\frac{3}{2}\frac{H'^2}{G'^2}\Big)\frac{a^3}{r^3}\cos(2g+2f)\Big]+$$

$$+\frac{\mu^4 k_2^2}{L'^7}\frac{a^3}{r^3}\Big[\tfrac{3}{4}(1-\theta'^2)\Big\{\Big(\frac{3}{e}\frac{a^4}{r^4}\eta'^{-2}-\frac{3}{e}\frac{a}{r}\eta'^{-1}\Big)\cos f-\frac{9}{2}\frac{a^3}{r^3}+3\eta'^{-3}\Big\}-$$

$$-\tfrac{3}{4}(1-3\theta'^2)(1-\theta'^2)\Big\{\Big(\frac{-9a^3}{r^3}-\frac{2a}{r}\eta'^{-2}+3\eta'^{-3}\Big)\cos(2g+2f)+\Big[2\frac{a^4}{r^4}\eta'^2-$$

$$\frac{a^3}{r^3}-\Big(\frac{3}{2}\eta'^{-2}+\tfrac{1}{2}\eta'^{-1}\Big)\frac{a}{r}\eta'^{-3}\Big]\frac{1}{e}\cos(2g+f)+\Big[4\frac{a^4}{r^4}\eta'^2+\frac{a^3}{r^3}-\Big(\frac{3}{2}\eta'^{-2}+$$

$$+\frac{5}{2}\eta'^{-1}\Big)\frac{a}{r}\eta'^{-3}\Big]\frac{1}{e}\cos(2g+3f)-\tfrac{1}{2}\frac{a}{r}\eta'^{-2}\cos(2g+4f)-\frac{3}{2}\frac{a}{r}\eta'^{-2}\cos 2g\Big\}+$$

$$+\frac{9}{4}(1-\theta'^2)\Big\{\Big(-\frac{9}{4}\frac{a^3}{r^3}-\frac{2}{3}\frac{a}{r}\eta'^{-2}-\tfrac{1}{3}\eta'^{-4}\Big)+\frac{3}{2e}\Big(\frac{a^4}{r^4}\eta'^{2}-\frac{a}{r}\eta'^{-2}\Big)\cos f-$$

$$-\Big(\frac{4}{3}\frac{a}{r}\eta'^{-2}+\tfrac{1}{3}\eta'^{-4}\Big)\cos 2f-\Big(\frac{1}{4}\frac{a}{r}\eta'^{-2}-\tfrac{1}{3}\eta'^{-4}\Big)\cos(4g+2f)+$$

$$+\tfrac{1}{2}\Big(\frac{a^4}{r^4}\eta'^2-\tfrac{1}{3}\frac{a^3}{r^3}-\tfrac{1}{2}\frac{a}{r}\eta'^{-2}+\tfrac{1}{3}\eta'^{-4}\Big)\frac{1}{e}\cos(4g+3f)+\Big(\frac{5}{4}\frac{a^4}{r^4}\eta'^2+$$

$$+\tfrac{1}{2}\frac{a^3}{r^3}-\frac{5}{4}\frac{a}{r}\eta'^{-2}-\tfrac{1}{2}\eta'^{-4}\Big)\frac{1}{e}\cos(4g+5f)-\Big(\frac{9}{4}\frac{a^3}{r^3}+$$

$$+\frac{4}{3}\frac{a}{r}\eta'^{-2}+\frac{1}{3}\eta'^{-4}\Big)\cos(4g+4f)-\Big(\frac{5}{12}\frac{a}{r}\eta'^{-2}+\frac{1}{6}\eta'^{-4}\Big)\times$$

$$\times \mathrm{Cos}\left(4g+6f\right)\Big\}+\frac{3}{2}\,\theta'^2\,(1-\theta'^2)\,\{3\eta'^{-4}\,[\tfrac{1}{2}+\frac{2}{3}\,\mathrm{Cos}\,f+\tfrac{1}{2}\,\mathrm{Cos}\,(4g+4f)+$$

$$\frac{e}{2}\,\mathrm{Cos}\,(4g+3f)+\frac{e}{6}\,\mathrm{Cos}\,(4g+5f)]-3\eta'^{-4}[\mathrm{Cos}\,(2g+2f)+e\,\mathrm{Cos}\,(2g+f)+$$

$$\frac{e}{3}\,\mathrm{Cos}\,(2g+3f)\,]\,\}-\frac{3\mu^4k_2^2}{32g'^7}\Big[-5\,(1-2\theta'^2-7\theta'^4)+\eta'^2\,(5-18\theta'^2+5\theta'^4)+2e^2$$

$$(1-\theta'^2)\,(1-15\theta'^2)\,\mathrm{Cos}\,2g]+\frac{\mu^4k^4}{L'^7}\Big[\Big(\frac{3}{8}-\frac{15}{4}\theta'^2+\frac{35}{8}\theta'^4\Big)\Big(\frac{a^5}{r^5}-\frac{5}{2}\eta'^{-7}+\frac{3}{2}\eta'^{-5}\Big)+$$

$$+\left(\frac{-5}{6}+\frac{20}{3}\,\theta'^2-\frac{35}{6}\,\theta'^4\right)\Big\{\frac{a^5}{r^5}\,\mathrm{Cos}\,(2g+2f)-\left(\frac{3}{4}\eta'^{-7}-\frac{3}{4}\eta'^{-5}\right)\times$$

$$\mathrm{Cos}\,2g\}+\left(\frac{35}{24}-\frac{35}{12}\,\theta'^2+\frac{35}{24}\,\theta'^4\right)\frac{a^5}{r^5}\mathrm{Cos}\,(4g+4f)\,\Big] \qquad \ldots\ (9)$$

$$G=G'+\frac{\partial S_1}{\partial g}+\frac{\partial S_2}{\partial g}$$

$$=G'+\frac{\mu^2k_2}{G'^3}\left(\frac{3}{2}-\frac{3}{2}\theta'^2\right)\Big[\mathrm{Cos}\,(2g+2f)+e\,\mathrm{Cos}(2g+f)+\frac{e}{3}\mathrm{Cos}(2g+3f)]+$$

$$+ \frac{\mu^3{}_2 k_2^2 L'^2}{512 G'^9} \Big[-192 e^2 \eta'^2 (1-\theta'^2)(1-15\theta'^2)(f-l) \operatorname{Sin} 2g + 12 e^4 \times$$

$$(1-3\theta'^2)(1-\theta'^2) \times \operatorname{Cos}(4f-2g) + (1-\theta'^2) \{144 e^3 (1-3\theta'^2) \operatorname{Cos}(3f-2g) -$$

$$48 e^2 (1-3\theta'^2)(-15+2\eta'^2) \operatorname{Cos}(2f-2g) + 48 e (1-3\theta'^2)(41-15\eta'^2$$

$$-2\eta'^3) \operatorname{Cos}(f-2g) + \frac{96}{e} [(1-3\theta'^2)(27-9\eta'^3+5\eta'^5) - \eta'^2 (25-3\theta'^2) + 2\eta'^4 \times$$

$$(1+33\theta'^2)] \operatorname{Cos}(2g+f) + 384[-\eta'^2(1+7\theta'^2) + 4\eta'^4\theta'^2] \operatorname{Cos}(2g+2f) + \frac{32}{e}$$

$$[(1-3\theta'^2)(-81+27\eta'^3+\eta'^5) + 5\eta'^2(11-41\theta'^2) - 2\eta'^4(1-23\theta'^2)] \operatorname{Cos}(2g+3f)$$

$$+12 [3(1-3\theta'^2)(-87+16\eta'^3) + 2\eta'^2(51-161\theta'^2) - \eta'^4 (1-19\theta'^2)] \times$$

$$\operatorname{Cos}(2g+4f) + 48e(1-3\theta'^2)(-41+7\eta'^2+2\eta'^3) \operatorname{Cos}(2g+5f) + 16e^2 \times$$

$$(1-3\theta'^2)(-45+2\eta'^2) \operatorname{Cos}(2g+6f) - 144e^3(1-3\theta'^2) \operatorname{Cos}(2g+7f) - 12e^4$$

$$(1-3\theta'^2) \operatorname{Cos}(2g+8f) - 18e^4(1-\theta'^2) \operatorname{Cos}(2f-4g) - 216e^3(1-\theta'^2) \operatorname{Cos}(f-4g)$$

$$+72e(1-\theta'^2) \times (-41+29\eta'^2) \operatorname{Cos}(4g+f) + 6[-783(1-\theta'^2) +$$

$$2\eta'^2 (335-239\theta'^2) + \eta'^4(65-257\theta'^2)] \operatorname{Cos}(4g+2f) + \frac{48}{e} [81(1-\theta'^2) -$$

$$-8\eta'^2(15-17\theta'^2) + \eta'^4(17-65\theta'^2)] \operatorname{Cos}(4g+3f) + 48[-\eta'^2(51-91\theta'^2) +$$

$$+\eta'^4(11-27\theta'^2)] \times \operatorname{Cos}(4g+4f) + \frac{48}{e} [81(1-\theta'^2) - 8\eta'^2(15-17\theta'^2) +$$

$+\eta'^4(39-550\theta'^2)] \times \text{Cos}\,(4g+5f)+2\,[2349\,(1-\theta'^2)-2\eta'^2\,(861-893\theta'^2)+$

$\eta'^4\,(157-221\theta'^2)]\,\text{Cos}\,(4g+6f)+32e\,(1-\theta'^2)\,(41-33\eta'^2)\,\text{Cos}\,(4g+7f)+$

$+24e^2(1-\theta'^2)\,(45-4\eta'^2)\,\text{Cos}\,(4g+8f)+216e^3\,(1-\theta'^2)\,\text{Cos}\,(4g+9f)+$

$+18e^4\,(1-\theta'^2)\,\text{Cos}\,(4g+10f)]+\frac{\mu^4 k_4}{24G'^7}\,[-(1-8\theta'^2+7\theta'^4)\,\{-60\,(f-l)\times$

$\times\,\text{Sin}\,2g+30\,(4e+e^3)\,\text{Cos}\,(2g+f)+20\,(2+3e^2)\,\text{Cos}\,(2g+2f)+10\,(4e+e^3)$

$\text{Cos}\,(2g+3f)+15e^2\,\text{Cos}\,(2g+4f)+2e^3\,\text{Cos}\,(2g+5f)-10e^3\,\text{Cos}\,(f-2g)\}+$

$(1-2\theta'^2+\theta'^4)\,\{35e^3\text{Cos}(4g+f)+105e^2\text{Cos}(4g+2f)+35(4e+e^3)\text{Cos}(4g+3f)$

$+35\,(2+3e^2)\times\text{Cos}\,(4g+4f)+21\,(4e+e^3)\,\text{Cos}\,(4g+5f)+35e^3\,\times$

$$\text{Cos}\,(4g+6f)+5e^3\,\text{Cos}\,(4g+7f)] \qquad \ldots \quad (10)$$

$$H=H' \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots\ (11)$$

$$l=l'-\frac{\partial S_1}{\partial L'}-\frac{\partial S_2}{\partial L'}=l-\left(\frac{\partial S_1}{\partial L'}\right)-\frac{G'^2}{eL'^3}\left(\frac{\partial S_1}{\partial e}\right)-\left(\frac{\partial S_2}{\partial L'}\right)-\frac{G'^2}{eL'^3}\left(\frac{\partial S_2}{\partial e}\right)$$

$$=l'-\frac{\mu^2 k_2}{eL'^3G'}\left\{\left(-\tfrac{1}{2}+\frac{3}{2}\frac{H'^2}{G'^2}\right)\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\right.$$

$$\text{Sin}\,f+\tfrac{1}{2}\left(\frac{3}{2}-\frac{3}{2}\frac{H'^2}{G'^2}\right)\times\left[\left(\frac{-a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\text{Sin}\,(2g+f)+\right.$$

$$\left(\frac{a^2G'^2}{r^2L'^2}+\frac{a}{r}+\frac{1}{3}\right)\text{Sin}\,(2g+3f)]\}-\frac{\mu^4k_2^2L'}{512G'^9}[-96\{\eta'^4\,(5-18\theta'^2+5\theta'^4)\times$$

$$(f-l)+\frac{48}{e}\{9\,(11-30\theta'^2+27\theta'^4)+6\eta'^3\,(1+\eta'^2)\,(1-3\theta'^2)^2-\eta'^4\times$$

$$(53-130\theta'^2-11\theta'^4)\}\,\text{Sin}\,f+6\,\{87\,(11-30\theta'^2+27\theta'^4)+32\eta'^3\,(1-3\theta'^2)^2$$

$$-\eta'^4(93-242\theta'^2+77\theta'^4)\}\text{Sin}2f+8e\{41(11-30\theta'^2+27\theta'^4)+8\eta'^3(1-3\theta'^2)^2\}\times$$

$$\text{Sin}\,3f+24e^2\,\{5(11-30\theta'^2+27\theta'^4)\}\,\text{Sin}4f+24e^3(11-30\theta'^2+27\theta'^4)^2\,\text{Sin}\,f5$$

$$+2e^4(11-30\theta'^2+27\theta'^4)\,\mathrm{Sin}\,6f+(1-\theta'^2)\{-12e^4\,(1-3\theta'^2)\,\mathrm{Sin}\,(4f-2g)$$

$$-144e^3(1-3\theta'^2)\,\mathrm{Sin}\,(3f-2g)-720e^2(1-3\theta'^2)\,\mathrm{Sin}\,(2f-2g)+24e$$

$$(1-3\theta'^2)(-82-2\eta'^3)\,\mathrm{Sin}\,(f-2g)+\frac{48}{e}[(1-3\theta'^2)(54+9\eta'^3-15\eta'^5)$$

$$-4\eta'^4(1+33\theta'^2)]\,\mathrm{Sin}\,(2g+f)-1536\eta'^4\theta'^2\,\mathrm{Sin}\,(2g+2f)-\frac{16}{e}[3(1-3\theta'^2\times$$

$$(54+9\eta'^3+\eta'^5)-4\eta'^4(1-23\theta'^2)]\times\mathrm{Sin}\,(2g+3f)+6[-6(1-3\theta'^2)\times$$

$$(87+8\eta'^3)+2\eta'^4(1-19\theta'^2)]\mathrm{Sin}(2g+4f)-48e(1-3\theta'^2)(41+\eta'^3)\mathrm{Sin}(2g+5f)$$

$$-720e^2(1-3\theta'^2\;\;\mathrm{Sin}\,(2g+6f)+144e^3\,(3\theta'^2-1)\;\;\mathrm{Sin}\,(2g+7f)+$$

$$+12e^4(3\theta'^2-1)\,\mathrm{Sin}\,(2g+8f)+108e^3(1-\theta'^2)\times\mathrm{Sin}\,(f-4g)+9e^4\,(1-\theta'^2)$$

$$\mathrm{Sin}\,(2f-4g)-1476e(1-\theta'^2)\mathrm{Sin}(4g+f)+5[-783(1-\theta'^2)-\eta'^4(65-257\theta'^2)\times$$

$$\mathrm{Sin}\,(4g+2f)+\frac{24}{e}[-81(1-\theta'^2)-\eta'^4(65-257\theta'^2)\,\mathrm{Sin}\,(4g+2f)+\frac{24}{e}\times$$

$$[-81(1-\theta'^2)-\eta'^4(17-65\theta'^2)]\,\mathrm{Sin}\,(4g+3f)-24\eta'^4(11-27\theta'^2)\mathrm{Sin}(4g+4f)$$

$$+\frac{24}{e}[81(1-\theta'^2)-\eta'^4(39-55\theta'^2)]\,\mathrm{Sin}\,(4g+5f)+[2349(1-\theta'^2)-\eta'^4\times$$

$$(157-221\theta'^2)]\,\mathrm{Sin}\,(4g+6f)+1476e\,(1-\theta'^2)\,\mathrm{Sin}\,(4g+7f)+540e^2(1-\theta'^2)$$

$$\mathrm{Sin}\,(4g+8f)+108e^3(1-\theta'^2)\,\mathrm{Sin}(4g+9f)+9e^4(1-\theta'^2)\times\mathrm{Sin}\,(4g+10f)-$$

$$-\frac{G'^2}{eL'^3}\left(\frac{\partial S_2}{\partial e}\right) \qquad (12)$$

Where $\frac{\partial S_2}{\partial e}$ is given by

$$\frac{\partial S_2}{\partial e}=\frac{\mu^4 k_2^2 L'^9}{1024\, G'^9}\Big[-\frac{48}{e^2}\,[9\,(11-30\theta'^2+27\theta'^4)\;-\;8\eta'^2\;(17-38\theta'^2+11\theta'^4)-$$

$$-4\eta'^3\,(1-3\theta'^2)\;(3+\eta'^2)+\eta'^4(53-130\theta'^2-11\theta'^4)]\,\mathrm{Sin}\,f+8[41(11-30\theta'^2+$$

$$+27\theta'^4)-\eta'^2\,(191-390\theta'^2+207\theta'^4-8\eta'^3\,(1-3\theta'^2)^2]\;\mathrm{Sin}\;3f+48e\times$$

$$[5\,(11-30\theta'^2+27\theta'^4)-8\eta'^2)\;(1-\theta'^2)^2]\mathrm{Sin}4f+72e^2(11-30\theta'^2+27\theta'^4)\mathrm{Sin}\;5f$$

$$+8e^3\,(11-30\theta'^2+27\theta'^4)\;\mathrm{Sin}\;6f+(1-\theta'^2)\,\{384e\eta'^2\;\;(1-15\theta'^2)\;(f-l)\times$$

$$\mathrm{Cos}\;2g-48e^3\;(1-3\theta'^2)\;\mathrm{Sin}\;(4f-2g)-432e^2\,(1-3\theta'^2)\;\mathrm{Sin}\;\;(3f-2g)+$$

$$+96e\,(1-3\theta'^2)\;(-15+2\eta'^2)\;\mathrm{Sin}\;(2f-2g)+48\;[(1-3\theta'^2)\;\;(-41+15\eta'^2+$$

$$+2\eta'^3)]\times\mathrm{Sin}\;(f-2g)-\;\frac{96}{e^2}[(1-3\theta'^2)\;(27-9\eta'^3+5\eta'^5)-\eta'^2\;\;(25-3\theta'^2)+$$

$$+2\eta'^4\;\;(1+33\theta'^2)]\times\mathrm{Sin}\;\;(2g+f)-\frac{32}{e^2}\,[(1-3\theta'^2,(-81+27\eta'^3+\eta'^5)+5\eta'^2\times$$

$$(11-41\theta'^2)-2\eta'^4\;\;(1-23\theta'^2)]\times\mathrm{Sin}\;\;(2g+3f)+48(1-3\theta'^2)\;\;(-41+7\eta'^2+$$

$$+2\eta'^3)\;\mathrm{Sin}\;(5f+2g)+32e\;(1-3\theta'^2)\;\;(-45+2\eta'^2)\;\;\mathrm{Sin}\;\;(2g+6f)-432e^2$$

$$(1-\theta'^2)\mathrm{Sin}\;(2g+7f)-48e^3\,(1-3\theta'^2)\mathrm{Sin}(2g+8f)+36e^3\;(1-\theta'^2)\mathrm{Sin}(2f-4g)$$

$$+324\theta^2\;\;(1-e'^2)\;\;\mathrm{Sin}\;\;(f-4g)\;+\;36\;\;(1-\theta'^2)\;\;\;(-41+29\eta'^2)\;\mathrm{Sin}\;(4g+f)$$

$$-\frac{24}{e^2}\,[-81\;(1-\theta'^2)+16\eta'^2\,(4-\theta'^2)+\eta'^4\;\;\;(17-65\theta'^2)]\;\mathrm{Sin}\;\;\;(4g+3f\,)-$$

$$-\frac{24}{e^2}\;[81\;(1-\theta'^2)-8\eta'^2\;(15-17\eta'^2)+\eta'^4\;(39-55\theta'^2)]\;\mathrm{Sin}\;(4g+5f)+$$

$$+36\;(1-\theta'^2)\;(41-13\eta'^2)\;\;\mathrm{Sin}\;(4g+7f\,)+24e(1-\theta'^2)(45-4\eta'^2)\mathrm{Sin}(4g-8f\,)$$

$$+324e^2\;\;(1-\theta'^2)\;\;\;\mathrm{Sin}\;\;\;\;(4g+9f\;)\;+\;36e^3\;\;(1-\theta'^2)\;\;\;\;\mathrm{Sin}\;\;(\;4g+10\,f\,)\,\}]\;+$$

$$+[96\;\;[-5\eta'^2\;\;(1-2\theta'^2-7\theta'^4)+\eta'^4\;\;\;(5-18\theta'^2+5\theta'^4)]+192e^2\eta'^2\,(1-\theta'^2)$$

$(1-15\theta'^2)$ Cos $2g+\frac{48}{e}$ $[9(11-30\theta'^2+27\theta'^4)-8\eta'^2(17-38\theta'^2+11\theta'^4)-$

$-4\eta'^3(1-3\theta'^2)(3+\eta'^2)+\eta'^4(53-130\theta'^2-11\theta'^4)]$ Cos $f+12$ $[87(11-30\theta'^2$

$+27\theta'^4)-2\eta'^2(389-834\theta'^2+437\theta'^4)-64\eta'^1(1-3\theta'^2)^2+\eta'^4(93-242\theta'^2+$

$+77\theta'^4)]$ Cos $2f+24e$ $[41(11-30\theta'^2+27\theta'^4)$ $\eta'^2(191-390\theta'^2+207\theta'^4)-$

$8\eta'^3(1-3\theta'^2)^2]$ Cos $3f+96e^2$ $[5(11-30\theta'^2++27\theta'^4)-8\eta'^2(1-\theta'^2)^2]$

Cos $4f+120e^3(11-30\theta'^2+27\theta'^4)$ Cos $5f+12e^4(11-30\theta'^2+27\theta'^4)$ Cos $6f$

$+(1-\theta'^2)\{-48e^4(1-3\theta'^2)$ Cos $(4g-2f)-432e^3(1-\theta'^2)$ Cos $(3f-2g)+$

$+196e^2(1-3\theta'^2)(-15+2\eta'^2)$ Cos $(2f-2g)+48e$ $(1-3\theta'^2)(41+15\eta'^2+$

$+2\eta'^3)$ Cos $(f-2g)+\frac{96}{e}[(1-3\theta'^2)(27-9\eta'^2+5\eta'^3)-\eta'^2(25-3\theta'^2)+$

$+2\eta'^4(1+33\theta'^4)]$ Cos $(2g+f)+768$ $[-\eta'^2(1+7\theta'^2)+4\eta'^4\theta'^2]\times$ Cos $(2g$

$+2f)+\frac{96}{e}[(1-3\theta'^2)(-81+27\eta'^2+\eta'^3)+5\eta'^2(11-41\theta'^2)-2\eta'^4\times$

$(1-23\theta'^2)]$ Cos$(2g+4f)+48$ $[3(1-3\theta'^2)(-87+16\eta'^3)+2\eta'^2(51-161\theta'^2)$

$-\eta'^4(1-19\theta'^2)]$ Cos $(2g+4f)-240e$ $[(1-3\theta'^2)(-41+7\eta'^2+2\eta'^3)]\times$

Cos $(2g+5f)+96e^2(1-3\theta'^2)(-45+2\eta'^2)$ Cos$(2g+6f)-1008e^3(1-3\theta'^2)\times$

Cos $(2g-7f)-96e^4(1-3\theta'^2)$ Cos $(2g+8f)+18e^4$ $(1-\theta'^2)$Cos $(2f-4g)$

$+168e^3(1-\theta'^2)$ Cos $(f-4g)+36e(1-\theta'^2)(-41+29\eta'^2)$ Cos $(4g+f)+6]$

$-783(1-\theta'^2)+2\eta'^2+(335-239\theta'^2)+\eta'^4(65-257\theta'^2)]$ Cos $(4g+2f)+\frac{72}{e}\times$

$[-81(1-\theta'^2)+16\eta'^2+(4-\theta'^2)+\eta'^4(17-65\theta'^2)]$ Cos $(4g+3f)+96[-\eta'^2$

$(51-91\theta'^2)+\eta'^4(11-27\theta'^2)]$ Cos $(4g+4f)+120/e$ $[81(1-\theta'^2)-8\eta'^2(15-$

$17\theta'^2)+\eta'^4(39-55\theta'^2)]$ Cos $(4g+5f)+6$ $[2349(1-\theta'^2)-2\eta'^2(861-893\theta'^2)$

$+\eta'^4(157-221\theta'^2)]$ Cos $(4g+6f)+252e(1-\theta'^2)(41-13\eta'^2)$ Cos $(4g+7f)$

$+96e^2(1-\theta'^2)(45-4\eta'^2)$ Cos $(4g+8f)+972e^3(1-\theta'^2)$ Cos $(4g+9f)+$

$+90e^4(1-\theta'^2)$ Cos $(4g+10f)\}]\left(\frac{a}{r}+\frac{L'^2}{G'^2}\right)$ Sin $f+\mu^4k_1/96G'^7$ $[2\times$

$(3-30\theta'^2+35\theta'^4)\{36e(f-l)+9(4+3e^2)\text{ Sin } f+18e \text{ Sin } 2f+3e^2 \text{ Sin } 3f\}-$
$-2(1-8\theta'^2+7\theta'^4)\{120e(f-l)\text{ Cos } 2g+30(4-3e^2)\text{ Sin }(2g+f)+$
$+120e\text{ Sin }(2g+2f)+10(4+3e^2)\text{ Sin }(2g+3f)+30e^2\text{ Sin }(f-2g)+$
$+30e\text{ Sin }(2g+4f)+6e^2\text{Sin }(2g+5f)\}+(-2\theta'^2+\theta'^4)\{105e^2\text{ Sin }(4g+f)+$
$+210e\text{ Sin }(4g+2f)+35(4+3e^2)\text{ Sin }(4g+3f)+210e\text{ Sin }(4g+4f)+$
$+21(4+3e^2)\text{ Sin }(4g+5f)+70e\text{ Sin }(4g+6f)+15e^2\text{ Sin }(4g+7f)\}+$
$+\mu^4k_4/96G'^7[2(3-30\theta'^2+35\theta'^4)\times\{6(2+3e^2)+9(4e+e^3)\text{ Cos } f+8e^2\times$
$\text{Cos}2f+3e^3\text{Cos } 3f\}-2(1-8\theta'^2+7\theta'^4)\{60e^2\text{Cos}2g+30(4e+e^3)\text{ Cos}(2g+f)$
$+40(2+3e^2)\times\text{Cos }(2g+2f)+30(4e+e^3)\text{ Cos }(2g+3f)+60e^2\times$
$\text{Cos }(2g+4f)+10e^3\text{ Cos }(2g+5f)+10e^3\text{ Cos }(f-2g)\}+(1-2\theta'^2+\theta'^4)\times$
$\{35e^3\text{ Cos }(4g+f)+210e^2\text{ Cos }(4g+2f)+105(4e+e^3)\text{ Cos }(4g+3f)+$
$+140(2+3e^2)\text{Cos }(4g+4f)+105(4e+e^3)\text{ Cos}(4g+5f)+210e^2\text{Cos}(4g+6f)$

$$+35e^3\text{ Cos }(4g+7f)\}]\times\left(\frac{a}{r}+\frac{L'^2}{G'^2}\right)\text{ Sin } f \qquad (13)$$

$$g=g'-\frac{\partial S_1}{\partial G'}-\frac{\partial S_2}{\partial G'}=g'-\left(\frac{\partial S'}{\partial G'}\right)+\frac{G'}{eL'^2}\left(\frac{\partial S_1}{\partial e}-\right)\left(\frac{\partial S_2}{\partial G'}\right)+\frac{G'}{eL'^2}\left(\frac{\partial S^2}{\partial e}\right)$$

$$=g'-\mu^2k_2/G'^4\left\{\left(-\frac{3}{2}+\frac{15}{2}\frac{H'^2}{G'^2}\right)\left(f-l+e\sin f\right)+\left(\frac{9}{2}-\frac{15}{2}\frac{H'^2}{G'^2}\right)\right.$$

$$\left[\tfrac{1}{2}\text{ Sin }(2g+2f)+\frac{e}{2}\text{ Sin }(2g+f)+\frac{e}{6}\text{ Sin }(2g+3f)\right]\Big\}+\mu^2k_2/eL'^2G'^2$$

$$\left\{\left(-\tfrac{1}{2}+\frac{3}{2}\frac{H'^2}{G'^2}\right)\left(\frac{a'}{r'}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\text{Sin } f+\tfrac{1}{2}\left(\frac{3}{2}-\frac{3}{2}\frac{H'^2}{G'^2}\right)\times\right.$$

$$\left[\left(\frac{-a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\text{Sin }(2g+f)+\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\text{ Sin }(2g+3f)\right]\Big\}$$

$+\mu^4k_2{}^2L'^2/1024\ G'^{10}[96\{-5\eta'^2(7-18\theta'^2-77\theta'^4)+\eta'^4(25-126\theta'^2+45\theta'^4\}$
$(f-l)+48/e[27(33-110\theta'^2+117\theta'^4)-8\eta'^2(119-342\theta'^2+121\theta'^4)-72\eta'^3\times$
$(1-8\theta'^2+15\theta'^4)-16\eta'^5(1-9\theta'^2+18\theta'^4)+\eta'^4(265-910\theta'^2-99\theta'^4)]\text{ Sin } f$
$+6[261(33-110\theta'^2+117\theta'^4)-2\eta'^2(2723-7506\theta'^2+4807\theta'^4)-384\eta'^3\times$
$(1-8\theta'^2+15\theta'^4)+\eta'^2(465-1694\theta'^2+693\theta'^4)]\text{ Sin } f+8e[13(33-$
$110\theta'^2+117'^4)-\eta'^2(1337-3510\theta'^2+2277\theta'^4)-48\eta'^3(1-8\theta'^2+1\theta'^4)]$
$\text{Sin } 3f+24e^2[15(33-110\theta'^2+117\theta'^4)-8\eta'^2(7-18\theta'^2+11\theta'^4)]\text{ Sin } 4f+$

$$
\begin{aligned}
& 72e^{3}\,(33-110\theta'^{2}+117\theta'^{4})\ \mathrm{Sin}\ 5f+6e^{4}\,(33-110\theta'^{2}+117\theta'^{4})\ \mathrm{Sin}\ 6f+ \\
& 192e^{2}\,\eta'^{2}\,(7-144\theta'^{2}+165\theta'^{4})\,(f-l)\ \mathrm{Cos}\ 2g+48e^{2}\,[-15\,(9-44\theta'^{2}+39\theta'^{4}) \\
& +2\eta'^{2}\,(7-36\theta'^{2}+33\theta'^{4})]\ \mathrm{Sin}\ (2f-2g)-12e^{4}\,(9-44\theta'^{2}+39\theta'^{4})\ \mathrm{Sin}\ (4f- \\
& 2g)-144e^{3}\,(9-44\theta'^{2}+39\theta'^{4})\ \mathrm{Sin}\ (3f-2g)+48e\ [-41\ (9-44\theta'^{2}+39\theta'^{4})+ \\
& 15\eta'^{2}\ (7-36\theta'^{2}+33\theta'^{4})+4\eta'^{3}\ (3-16\theta'^{2}+15\theta'^{4})]\ \mathrm{Sin}\ (f-2g)+96/e\ [27 \\
& (9-44\theta'^{2}+39\theta'^{4})-\eta'^{2}\ (175-252\theta'^{2}+33\theta'^{4}))-18\eta'^{3}\ (3-16\theta'^{2}+15\theta'^{4})+ \\
& 2\eta'^{4}\,(5+224\theta'^{2}-29\theta'^{4})+20\eta'^{5}\ (7-6\theta'^{2}+6\theta'^{4})]\ \mathrm{Sin}\ (2g+f\ )+384\ [-\eta'^{2} \\
& (7+54\theta'^{2}-77\theta'^{4})+4\eta'^{4}\theta'^{2}\ (7-9\theta'^{2})]\ \mathrm{Sin}\ (2g+2f\ )+32/e\ [-81\ (9-44\theta'^{2} \\
& +39\theta'^{4})+5\eta'^{2}\ (77-468\theta'^{2}+451\theta'^{4})+54\eta'^{3}\ (4-16\theta'^{2}+15\theta'^{4})-2\eta'^{4}\ (5- \\
& 188\theta'^{2}+207\theta'^{4})+4\eta'^{5}\ (1-6\theta'^{2}+6\theta'^{4})]\ \mathrm{Sin}\ (2g+3f\ )+12\ [-261\ (9-44\theta'^{2} \\
& +39\theta'^{4})+2\eta'^{2}\ (357-1908\theta'^{2}+1771\theta'^{4})+96\eta'^{3}\ (3-16\theta'^{2}+15\theta'^{4})-\eta'^{4}\ (5- \\
& 140\theta'^{2}+171\theta'^{4})]\ \mathrm{Sin}\ (2g+4f)+48e\ [-41\ (9-44\theta'^{2}+39\theta'^{4})+7\eta'^{2}\ (7- \\
& 36\theta'^{2}+33\theta'^{4})+4\eta'^{3}(3-16\theta'^{2}+15\theta'^{4})]\times\mathrm{Sin}\ (2g+5f)+16e^{2}[-45(9-44\theta'^{2}\times \\
& +39\theta'^{4})+2\eta'^{2}\ (7-36\theta'^{2}+33\theta'^{4})]\times\mathrm{Sin}\ (2g+6f)-144e^{3}(9-44\theta'^{2}+39\theta'^{4})\times \\
& \mathrm{Sin}\ (2g+7f\ )-12e^{4}\ (9-44\theta'^{2}+39\theta'^{4})\ \mathrm{Sin}\ (2g+8f\ )+9e^{4}\ (9-22\theta'^{2}+13\theta'^{4}) \\
& \mathrm{Sin}\ (2f-2g)+108e^{3}\ (9-22\theta'^{2}+13\theta'^{4})\ \mathrm{Sin}\ (f-4g)+36e\ [-41\ (9-22\theta'^{2}+ \\
& 13\theta'^{4})+29\eta'^{2}\ (7-18\theta'^{2}+11\theta'^{4})]\ \mathrm{Sin}\ (4g+f\ )+3\ [-783\ (9-22\theta'^{2}+13\theta'^{4}) \\
& +2\eta'^{2}(2345-5166\theta'^{2}+2629\theta'^{4})+\eta'^{4}(325-2254\theta'^{2}+2313\theta'^{4})]\ \mathrm{Sin}(4g+2f\ ) \\
& +\frac{24}{e}\,[-81\ (9-22\theta'^{2}+13\theta'^{4})+16\eta'^{2}\ (28-45\theta'^{2}-11\theta'^{4})+\eta'^{4}\ (85- \\
& 574\theta'^{2}+585\theta'^{4})]\ \mathrm{Sin}\ (4g+3f\ )+24\ [-\eta'^{2}\ (357-1278\theta'^{2}+1001\theta'^{4})+\eta'^{4}\times \\
& (55-266\theta'^{2}+243\theta'^{4})]\ \mathrm{Sin}\ (4g+4f\ )+24/e\ [81\ (9-22\theta'^{2}+13\theta'^{4})-8\eta'^{2}\times \\
& (105-288\theta'^{2}+187\theta'^{4})+\eta'^{4}\ (195-658\theta'^{2}+495\theta'^{4})]\ \mathrm{Sin}\ (4g+5f\ )+[2349 \\
& (9-22\theta'^{2}+13\theta'^{4})-2\eta'^{4}\ (6027-15786\theta'^{2}+9823\theta'^{4})+\eta'^{4}\ (785-2046\theta'^{2}+ \\
& 1989\theta'^{4})]\mathrm{Sin}(4g+6f)+36e\ [41(9-22\theta'^{2}+13\theta'^{4})-13\eta'^{2}(7-18\theta'^{2}+11\theta'^{4})\times \\
& [(\mathrm{Sin}\ (4g+7f\ )]+12e^{2}\ [45\ (9-22\theta'^{2}+13\theta'^{4})-4\eta'^{2}\ (7-18\theta'^{2}+11\theta'^{4})] \\
& \mathrm{Sin}(4g+8f\ )+108e^{3}\ (9-22\theta'^{2}+13\theta'^{4}).\mathrm{Sin}\ (4g+9f\ )+9e^{4}(9-22\theta'^{2}+13\theta'^{4} \\
& \mathrm{Sin}(4g+10f\ )]-\mu^{4}k_{4}/96g'^{8}\ [2\ (21-270\theta'^{2}+385\theta'^{4})\times\{6\ (f-l)\ (2+3e^{2})+9 \\
& (4e+e^{3})\ \mathrm{Sin}\ f+9e^{2}\ \mathrm{Sin}\ 2f+e^{3}\ \mathrm{Sin}\ 3f\}-2\ (7-72\theta'^{2}+77\theta'^{4})\ \{60\ (f-l)\ e^{2} \\
& \mathrm{Cos}\ 2g+30\ (4e+e^{3})\ \mathrm{Sin}\ (2g+f\ )+20\ (2+3e^{2})\ \mathrm{Sin}\ (2g+2f\ )+10\ (4e+ \\
& e^{3})\ \mathrm{Sin}\ (2g+3f\ )+15e^{2}\ \mathrm{Sin}\ (2g+4f\ )+2e^{3}\ \mathrm{Sin}\ (4g+5f\ )+10e^{4}\ \mathrm{Sin}\ (f- \\
& 2g)\}+(7-18\theta'^{2}+11\theta'^{4})\ \{35e^{3}\ \mathrm{Sin}\ (4g+f\ )+105e^{2}\ \mathrm{Sin}\ (4g+2f)+35\ (4e+ \\
& e^{3})\ \mathrm{Sin}\ (2g+3f\ )+35\ (2+3e^{2})\ \mathrm{Sin}\ (4g+4f\ )+21\ (4e+e^{3})\ \mathrm{Sin}\ (4g+5f\ )+ \\
& 35e^{2}\ \mathrm{Sin}\ (4g+6f\ )+5e^{3}\ \mathrm{Sin}\ (4g+7f\ )]+\frac{G'}{e\,L'^{2}}\left(\frac{\partial S_{2}}{\partial e}\right) \qquad (14)
\end{aligned}
$$

Where $\left(\frac{\partial S_2}{\partial e}\right)$ is given by (13)

$$h = h' - \frac{\partial S_1}{\partial H'} - \frac{\partial S_2}{\partial H'}$$

$$= h' - 3\mu^2 k_2/G'^4 \left[f - l + e \sin f - \tfrac{1}{2} \sin(2g+2f) - \frac{e}{2}\sin(2g+f) - \frac{e}{6}\times \sin(2g+3f)\right]\frac{H'}{G'} - \mu^4 k_2^2 L'^2/256 G'^{10} [\{5\eta'^2(1+7\theta'^2) - \eta'^4(9-5\theta'^2)\}\times (f-l) - 192 e^2\eta'^2(8-15\theta'^2)(f-l)\cos 2g + 48/e.[-9(15-27\theta'^2)+8\eta'^2(19-11\theta'^2)+12\eta'^3(3+\eta'^2)(1-3\theta'^2)-\eta'^4(65+11\theta'^2)]\sin f - 6[87(15-27\theta'^2) - 2\eta'^2(417-437\theta'^2) - 192\eta'^3(1-3\theta'^2)+\eta'^4(121-77\theta'^2)]\times\sin 2f + 8e\times[-41(15-27\theta'^2)+\eta'^2(195-207\theta'^2)+24\eta'^3(1-3\theta'^2)]\sin 3f + 24e^2[-5\times(15-27\theta'^2)+8\eta'^2(1-8\theta'^2)]\sin 4f + 24e^3(15-27\theta'^2)\sin 5f - 2e^4 15 - 27\theta'^2) \sin 6f + 12e^4(2-3\theta'^2)\sin(4f-2g) + 144e^3(2-3\theta'^2)\times \sin(3f-2g) + 48e^2(15-2\eta'^2)(2-3\theta'^2)\sin(2f-2g)+48e(2-3\theta'^2)+(41-15\eta'^2-2\eta'^3)\sin(f-2g)+96/e[(2-3\theta'^2)(-27+9\eta'^3-5\eta'^5)+\eta'^2(14-3\theta'^2) - 2\eta'^4(16-33\theta'^2)]\sin(2g+f) + 384[-\eta'^2(3-7\theta'^2)+2\eta'^4(1-2\theta'^2)]\sin(2g+2f)+32/e[(2-3\theta'^2)(81-27\eta'^3-\eta'^5)-5\eta'^2(26-41\theta'^2)+2\eta'^4(12-23\theta'^2)]\sin(2g+3f)+12[3(2-3\theta'^2)+(87-16\eta'^3)-2\eta'^2(106-161\theta'^2)+\eta'^4(10-19\theta'^2)]\sin(2g+4f)+48e(2-3\theta'^2)\div(41-7\eta'^2-2\eta'^2)\sin(2g+5f)+16e^2(2-3\theta'^2)(45-2\eta'^2)\sin(2g+6f)+144e^3(2-3\theta'^2)\sin(2g+7f)+12e^4(2-3\theta'^2)\sin(2g+1f)-108e^3(1-\theta'^2)\times\sin(f-4g) - 9e^4(1-\theta'^2)\times\sin(2f-4g)+36e(1-\theta'^2)(41-29\eta'^2)\sin(4g+f)+3[783(1-\theta'^2)-2\eta'^2(287-239\theta'^2)-\eta'^4(161-257\theta'^2)]\sin(4g+2f)+24/e[81(1-\theta'^2)-8\eta'^2(5-2\theta'^2)-\eta'^4(41-65\eta'^2)]\sin(4g+3f)+24[\eta'^2(71-91\theta'^2)-\eta'^4(19-27\theta'^2)]\sin(4g+4f)+24/e[-81(1-\theta'^2)+8\eta'^2(16-17\theta'^2)-\eta'^4(47-55\theta'^2)]\times\sin(4g+5f)+[-2349(1-\theta'^2)+2\eta'^2(877-893\theta'^2)-\eta'^4(289-251\theta'^2)]\times\sin(4g+6f)-36e(1-\theta'^2)(41-13\eta'^2)\sin(4g+7f)-12e^2(1-\theta'^2)(45-4\eta'^2)\sin(4g+8f)-108e^3(1-\theta'^2)\sin(4g+9f)-9e^4(1-\theta'^2)\sin(4g+10f)]\frac{H'}{G'} + \mu^4 k_4/24G'^8[-2(15-35\theta'^2)\{6(f-l)(2+3e^2)+9(4e+e^3)\sin f$$

$+9e^2$ Sin $2f+e^3$ Sin $3f\}+2\,(4-7\theta'^2)\,\{60\,(f-l)\,e^2$ Cos $2g+30\,(4e+e^3)$ Sin $(2g+f)+20\,(2+3e^2)$ Sin $(2g+2f)+10\,(4e+e^3)$ Sin $(2g+3f)+15e^2\times$ Sin $(2g+4f)+2e^3$Sin$(2g+5f)+10e^3$ Sin $(f-2g)\}-(1-\theta'^2)\{35e^3$ Sin $(4g+$ $+f)+105e^2$ Sin $(4g+2f)+35\,\{(4e+e^3)$ Sin $(4g+3f)+35\,(2+3e^2)$ Sin $(4g+4f)+21\,(4e+e^3)$ Sin $(4g+5f)+35e^2$ Sin $(4g+6f)+5e^3$

$$\text{Sin }(4g+7f)\,\frac{H'}{G'} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (15)$$

In a similar way the long period and secular perturbations may be calculated. There appears e as a divisor in the Delaunay variables but if the expressions under bracket be expanded, they will cancel out. Moreover, in a little different but otherwise same problem Yoshide Kozai in his paper [4] has calculated these perturbations. It is expected that the calculation will be only mechanical. Taking into consideration the mechanical value, we drop the idea of the calculation of long period perturbations and secular perturbations.

6. Calculation of Polar Coordinates.

The function S_2 can be thought of in various ways. The usual concept is to consider it as the function of the elements L', G', H', l', g', h'. But it may be considered as a function of $\dot{r}'$, G', H', r', u', h' where r and u are the radius vector and the argument of the latitude.

The first order perturbation in $\dot{r}$, r and u has been obtained by Izsak [3,1963] by the formulas :

$$\dot{r}_1=\frac{\partial S_1}{\partial r},\ r_1=-\frac{\partial S_1}{\partial \dot{r}},\ u_1=-\frac{\partial S_2}{\partial G}$$

In similar manner the second order perturbations in coordinates r, $\dot{r}$ and u are given by

$$\dot{r}_2=\frac{\partial S_2}{\partial r},\ \ r_2=-\frac{\partial S_2}{\partial \dot{r}},\ u_2=-\frac{\partial S_2}{\partial G}$$

Now putting $f-l=\gamma$ which is called the equation of centre and introducing the argument of latitude $u=g+f$ and using the equations of Keplerian motion.

$$e\ \cos f = G^2/\mu r\ -1 \text{ and } e \sin f = G\ \dot{r}/\mu$$

the determining function S_2 can be transformed into new variables.

Then these three coordinates are given by

$$\dot{r}_2 = \frac{\partial S_2}{\partial r} = \left(\frac{\partial S_2}{\partial \gamma}\right)\frac{\partial \gamma}{\partial r} + \left(\frac{\partial S_2}{\partial e}\right)\frac{\partial e}{\partial r} + \left(\frac{\partial S_2}{\partial f}\right)\frac{\partial f}{\partial r}$$

$$= \mu^{\frac{1}{2}} / 3072 p^{9/2}\Big[\ e \operatorname{Sin} f\{(1-e^2)^{\frac{1}{2}} + (1+e \operatorname{Cos} f)^2 / \left(1+\sqrt{1-e^2}\right)\}\times$$

$$\Big(3k_2{}^2[96\{-5\ (1-2\theta'^2-7\theta'^4)+\eta'^2\ (5-18\theta'^2+5\theta'^4)\}+192e^2\ (1-16\theta'^2+$$

$$15\theta'^4) \operatorname{Cos}(2u-2f)]+32k_4\ [12\ (2+3e^2)\ (3-30\theta'^2+35\theta'^4)-120e^2\ (1-8\theta'^2$$

$$+7\theta'^4) \operatorname{Cos}(2u-2f)\ ]\Big) - \operatorname{Cos} f(1+e \operatorname{Cos} f)^2 \times \Big(A\Big) + \frac{\operatorname{Sin} f}{e}\ (1+$$

$$e \operatorname{Cos} f)^2 \times (B) \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (16)$$

Where (A) and (B) are given by

$$(A) = 3k_2{}^2\ \{192\gamma e\ [2\ (1-16\theta'^2+15\theta'^4) \operatorname{Cos}(2u-2f) - (5-18\theta'^2+5\theta'^4)]-48\times$$
$$[9\eta'^{-2}\ (e^{-2}-2)\ (11-30\theta'^2+27\theta'^4)-4\eta' e^{-2}\ (3+\eta'^2)(1-3\theta'^2)^2 - 12(\eta'^{-1}+\eta')$$
$$\times(1-3\theta'^2)^2-8e^{-2}\ (17-38\theta'^2+11\theta'^4)+(2+\eta'^2 e^{-2})\ (53-130\theta'^2-11\theta'^4)]\times$$
$$\operatorname{Sin} f+6e\ [174\eta'^{-2}\ (11-30\theta'^2+27\theta'^4)+64\eta'^{-1}\ (1-3\theta'^2)^2-2\ (93-242\theta'^2+$$
$$77\theta'^4)]\ \operatorname{Sin} 2f+8\ [41\eta'^{-2}\ (1+2e^2)\ (11-30\theta'^2+27\theta'^4)-8\ (1-3\theta'^2)^2\ (\eta'-$$
$$e^2\eta'^{-1})-(191-390\theta'^2+207\theta'^4)] \operatorname{Sin} 3f+48e\ [5\eta'^{-2}\ (1+2e^2)\ (11+30\theta'^2+$$
$$27\theta'^4)-3\ (1-\theta'^2)^2]\ \text{Siu } 4f-24e\eta'^{-2}\ (3+2e^2)\ (11-30'^2+27\theta'^4)\ \operatorname{Sin} 5f+$$
$$4e^3\eta'^{-2}\ (2+e^2)\ (11-30\theta'^2+27\theta'^4)\times \operatorname{Sin} 6f-24e^3\eta'^{-2}\ (2-e^2)(1-4\theta'^2+3\theta'^4)$$
$$\operatorname{Sin}(6f-2u)-144e^2\eta'^{-}\quad (1-4\theta'^2+3\theta'^4)\times \operatorname{Sin}\ (5f-2u)-96e\ \ (1-4\theta'^2+$$
$$3\theta'^4)\ [-15\eta'^{-2}\ (1+e^2)+2] \operatorname{Sin}(4f-2u)+48\ (1-4\theta'^2+3\theta'^4)\ [-41\eta'^{-2}\ (1+$$
$$2e^2)-2\eta'^{-1}e^2+15] \operatorname{Sin}(3f-2u)-96\ [(1-4\theta'^2+3\theta'^4)\ \ [27\eta'^{-2}\ (e^{-2}-2)-9\times$$
$$(\eta'^{-1}+\eta' e^{-2})+15\eta'+5\eta'^3 e^{-2}]-e^{-2}\ \ (25-28\theta'^2+3\theta'^4)+2\ \ (\eta'^2 e^{-2}+2)\ \ (1+$$

$32\theta'^2-33\theta'^4)]\ \mathrm{Sin}\ (2u-f)-3072e\theta'^2\ (1-\theta'^2)\ \mathrm{Sin}\ 2u-32\ [\ (1-4\theta'^2+3\theta'^4)$
$\{-81\eta'^{-2}\ (e^{-2}-2)+27\ (\eta'^{-1}+\eta'e^{-2})+3\eta'+\eta'^3e^{-2}\}+5e^{-2}\ (11-52\theta'^2+41\theta'^4)$
$-2\eta'^2\ (1-24\theta'^2+27\theta'^4)\ e^{-2}+2)]\ \mathrm{Sin}\ (2u+f)-12e\ [3\ (1-4\theta'^2+3\theta'^4)$
$(174\eta'^{-2}+16\eta'^{-1})-2\ (1-20\theta'^2+19\theta'^4)]\ \mathrm{Sin}\ (2u+2f)+48\ (1-4\theta'f+3\theta'^4)$
$[-41\eta'^{-2}\ (1+e^2)+2\ (\eta'-e^2\eta'^{-1})+7]\ \mathrm{Sin}\ (2u+3f)+32e\ (1-4\theta'^2+3\theta'^4)\times$
$[-45\eta'^{-2}\ (1+e^2)+2]\ \mathrm{Sin}\ (2u+4f)-144e^2\eta'^{-2}\ (3+2e^2)\ (1-4\theta'^2+3\theta'^4)$
$\mathrm{Sin}\ (2u+5f)-24e^3\eta'^{-2}\ (2+e^2)\ (1-4\theta'^2+3\theta'^4)\mathrm{Sin}(2u+6f)+108e^2\eta'^{-2}\ (3+$
$2e^2)\ (1-\theta'^2)^2\ \mathrm{Sin}\ (5f-4u)+18e^3\eta'^{-2}\ (2+e^2)\ (1-\theta'^2)\ \mathrm{Sin}(6f-4u)+36\ (1-$
$\theta'^2)^2\ [-41\eta'^{-2}\ (1+2e^2)+29]\ \mathrm{Sin}\ (4u-3f)-3e\ [1566\eta'^{-2}\ (1-\theta'^2)^2+2\ (65-$
$322\theta'^2+257\theta'^4)]\ \mathrm{Sin}\ (4u-2f)-24\ (1-\theta'^2)^2\times 81\eta'^{-2}\ (2-e^{-2})+16e^{-2}\ (4-$
$5\theta'^2+\theta'^4)+(2+\eta'^2e^{-2})\ (17-82\theta'^2+65\theta'^4)]\ \mathrm{Sin}\ (4u-f)-18e\ (11-38\theta'^2+$
$27\theta'^4)\ \mathrm{Sin}\ 4u-24\ [81\eta'^{-2}\ (e^{-2}-2)\ (1-\theta'^2)^2-1e^{-2}\ (15-32\theta'^2+17\theta'^4)+$
$(2+\eta'^2e^{-2})\ (39-94\theta'^2+55'^4)]\ \mathrm{Sin}\ (4u+f)+e\ [4668\eta'^{-2}(1-\theta'^2)^2-(157-$
$378\theta'^2+221\theta'^4)]\ \mathrm{Sin}\ (4u+2f)+36(1-\theta'^2)^2\ [41\eta'^{-2}\ (1+2e^2)-13]\ \mathrm{Sin}\ (4u+$
$3f)+24e\ (1-\theta'^2)^2\ [45\eta'^{-2}\ (1+e^2)-4]\ \ \mathrm{Sin}\ (4u+4f)+108e^2\eta'^{-2}.(2+e^2)$
$(1-\theta'^2)^2\ \mathrm{Sin}\ (4u+5f)+18e^3\eta'^{-2}\ (2+e^2)\ (1-\theta'^2)^2\ \mathrm{Sin}\ (4u+6f)]+32k_4\times$
$\{2\ (3-30\theta'^2+35\theta'^4)\ ]\ 36\gamma e+9\ (4+3e^2)\times\mathrm{Sin}\ f+18e\ \mathrm{Sin}\ 2f+3e^2\ \mathrm{Sin}\ 3f)$
$-2\ (1-8\theta'^2+7\theta'^4)\ [120\gamma e\mathrm{Cos}(2u-2f)+30(4+3e^2)\times\mathrm{Sin}(2u-f)+120e\times$
$\mathrm{Sin}\ 2u+10\ (4+3e^2)\ \mathrm{Sin}\ (2u+f)+30e^2\ \mathrm{Sin}\ (2u+3f)+6e^2\ \mathrm{Sin}\ (2u+3f)$
$+10e^3\ \mathrm{Sin}\ (3f-2u)]+(1-\theta'^2)^2\ [105e^2\ \mathrm{Sin}\ (4u+3f)+210e\ \mathrm{Sin}\ (4u-2f)$
$+35\ (4+3e^2)\times\mathrm{Sin}\ (4u-f)+210e\ \mathrm{Sin}4u+21\ (4+3e^2)\ \mathrm{Sin}\ (4u+f)+70e\times$
$\mathrm{Sin}\ (4u+2f)+15e^2\ \mathrm{Sin}\ (4u+3f)]\}$

$(\mathrm{B})=3k_2^2[384e^2\gamma\ (1-16\theta'^2+15\theta'^4)\ \mathrm{Sin}\ (2u-2f)+48e\ [9\eta'^{-2}\ (11-30\theta'^2+$
$27\theta'^4)-8\ (17-38\theta'^2+11\theta'^4)-4\eta'\ (3+\eta'^2)\ (1-3\theta'^2)^2+\eta'^2\ (53-150\theta'^2-$
$11\theta'^4)]\ \mathrm{Cos}\ f+12[87\eta'^{-2}\ (11-30\theta'^2+27\theta'^4)-2\ \ (389-834\theta'^2+437\theta'^4)-$
$64\eta'\ (1-3\theta'^2)^2+\eta'^2\ [93-242\theta'^2+77\theta'^4)]\ \mathrm{Cos}\ 2f+24e[41\eta'^{-2}\ (11-30\theta'^2+$
$27\theta'^4)-(191-390\theta'^2+207\theta'^4)-8\eta'\ (1-3\theta'^2)^2]\ \ \mathrm{Cos}\ 3f+96e^2\ [5\eta'^{-2}\ (11-$
$30\theta'^2+27\theta'^4)-8\ (1-\theta'^2)^2]\ \mathrm{Cos}\ 4f+12e^3\eta'^{-2}\ (11-30\theta'^2+27\theta'^4)\ \mathrm{Cos}\ 5f+$
$12e^4\eta'^{-2}\ (11-30\theta'^2+27\theta'^4)\ \mathrm{Cos}\ 6f-72e^4\eta'^{-2}\ (1-4\theta'^2+3\theta'^4)\ \mathrm{Cos}\ \ (6f-2u)$
$-720e^3\eta'^{-2}\ (1-4\theta'^2+3\theta'^4)\ \mathrm{Cos}\ (5f-2u)+192e^2\ (-15\eta'^{-2}+2)\ \ (1-4\theta'^2+$
$3\theta'^4)\times\mathrm{Cos}\ (4f-2u)+144e\ (1-4\theta'^2+3\theta'^4)\ \ (-41\eta'^{-2}+15+2\eta')\ \ \mathrm{Cos}\ \ (3f-$
$2u)-96e[(1-4\theta'^2+3\theta'^4)\ (27\eta'^{-2}-9\eta'+5\eta'^3)-(25-28\theta'^2+3\theta'^4)+2\eta'^2\times$

$(1+32\theta'^2-33\theta'^4)]$ Cos $(2u-f)+32/e\ [(1-4\theta'^2+3\theta'^4)\ (-81\eta'^{-2}+27\eta'+\eta'^3)+5\ (11-52\theta'^2+41\theta'^4)-2\eta'^2\ (1-24\theta'^2+23\theta'^4)]$ Cos $(2u+f)+24\ [3\ (1-4\theta'^2+3\theta'^4)\ (-87\eta'^{-2}+16\eta')+2\ (51-212\theta'^2+161\theta'^4)-\eta'^2\ (1-20\theta'^2+19\theta'^4)]$ Cos $(2u+2f)+144e\ (1-4\theta'^2+3\theta'^4)\ (-41\eta'^{-2}+7+2\eta')\times$ Cos $(2u+3f)+64e^2\ (1-4\theta'^2+3\theta'^4)\ (-45\eta'^{-2}+2)$ Cos $(2u+4f)-720e^3\eta'^{-2}+(1-4\theta'^2+3\theta'^4)$ Cos $(2u+5f)-72e^4\eta'^{-2}\ (1-4\theta'^2+3\theta'^4)$ Cos $(2u+6f)+540e^3\eta'^{-2}\ (1-\theta'^2)^2$ Cos $(5f-4u)+54e^4\eta'^{-2}\ (1-\theta'^2)^2$ Cos $(6f-4u)-108e\times(1-\theta'^2)^2\ (-41\eta'^{-2}+29)$ Cos $(4u-3f)-6\ [-783\eta'^{-2}\ (1-\theta'^2)^2+2\ (335-574\theta'^2+239\theta'^4)+\eta'^2\ (65-322\theta'^2+257\theta'^4)]$ Cos $(4u-2f)-\frac{24}{e}\ [-81\eta'^{-2}\ (1-\theta'^2)^2+16\ (4-5\theta'^2+\theta'^4)+\eta'^2\ (17-82\theta'^2+65\theta'^4]$Cos $(4u-f)+24/e\times[81\eta'^{-2}(1-\theta'^2)^2-8(15-32\theta'^2+17\theta'^4)+\eta'^2(39-94\theta'^2+55\theta'^4)]Cos(4u+f)++2\ [2349\eta'^{-2}\ (1-\theta'^2)-2\ \ (861-1754\theta'^2+893\theta'^4)+\eta'^2\ (157-328\theta'^2+221\theta'^4)\times$Cos $(4u+2f)+108e\ (1-\theta'^2)^2\ (41\eta'^{-2}-13)$ Cos $(4u+3f+48e^2\ (1-\theta'^2)^2\times(45\eta'^{-2}-4)$ Cos $(4u+4f)+540e^3\eta'^{-2}\ (1-\theta'^2)^2$ Cos $(4u+5f)+54e^4\eta'^{-2}(1-\theta'^2)^2\times$Cos$(4u+6f)+32k_4\ [2\ (3-30\theta'^2+35\theta'^4)\{9(4e+e^3)Cosf+18e^2Cos2f+3e^3Cos3f\}-2\ (1-8\theta'^2+7\theta'^4)\ \{120\gamma e^2$ Sin $(2u-2f)-30\times(4e+e^3)$ Cos $(2u-f)+10\ (4e+e^3)$ Cos $(2u+f)+30e^2$ Cos $(2u+2f)+6e^3$ Cos $(2u+3f)+30e^3$Cos $(3f-2u)\}+(1-\theta'^2)^2\ \{-105e^3$ Cos $(4u-3f)-210e^2$ Cos $(4u-2f)-35\ (4e+e^3)$ Cos $(4u-f)+21\ (4e+e^3)$ Cos $(4u+f)+70e^2$ Cos $(4u+2f)+15e^3$ Cos $(4u+3f)\}\ ]$

$$r_2=-\frac{\partial S_2}{\partial r}=-\left(\frac{\partial S_2}{\partial \gamma}\right)\left(\frac{\partial \gamma}{\partial r}\right)-\left(\frac{\partial S_2}{\partial e}\right)\left(\frac{\partial e}{\partial r}\right)-\left(\frac{\partial S_2}{\partial f}\right)\left(\frac{\partial f}{\partial r}\right)$$

$$=-1/3072p^3\Big[\Big\{2(1-e^2)^{1/2}(1+e\ \mathrm{Cos}\ f)+e\ \mathrm{Cos}\ f/\ (1+\sqrt{1-e^2})\Big\}\times\Big(3k_2{}^2\Big)\ [96\times\{-5\ (1-2\theta'^2-7\theta'^4)+\eta'^2(5-18\theta'^2+5\theta'^4\}+192e^2\ (1-16\theta'^2+15\theta'^4)\times\times\mathrm{Cos}\ (2u-2f)]+3k_4\ [12\ (2+3e^2)\ (3-30\theta'^2+35\theta'^4)-120e^2\times(1-8\theta'^2+7\theta'^4)\ \mathrm{Cos}\ (2u-2f)]\Big)-\mathrm{Sin}\ f\times(\mathrm{A})+\mathrm{Cos}\ f/e\times(\mathrm{B})\qquad\ldots\qquad\ldots(17)$$

$$u_2 = -\frac{\partial S_2}{\partial G'} = -\left[\left(\frac{\partial S_2}{\partial G'}\right) + \left(\frac{\partial S_2}{\partial e}\right)\frac{\partial e}{\partial G'} + \left(\frac{\partial S_2}{\partial f}\right)\frac{\partial f}{\partial G'} + \left(\frac{\partial S_2}{\partial \gamma}\right)\frac{\partial \gamma}{\partial G'}\right]$$

$$= -\left[\left(\frac{\partial S_2}{\partial \gamma}\right)\frac{\partial \gamma}{\partial G'} + \left(\frac{\partial S_2}{\partial e}\right)\frac{\partial e}{\partial G'} + \left(\frac{\partial S_2}{\partial f}\right)\frac{\partial f}{\partial G'} + \frac{\partial S_2}{\partial G'}\right]$$

ie $u_2 = -1/3072p^4\ [-e\ \mathrm{Sin}\, f\cdot\{(2+e\ \mathrm{Cos}\, f)/(1+\sqrt{1-e^2})\}\times(3k_2^2\ [96\{-5(1-2\theta'^2-7\theta'^4)+\eta'^2(5-18\theta'^2+5\theta'^4)\}+192e^2(1-16\theta'^2+15\theta'^4)\ \mathrm{Cos}\,(2u-2f)$
$+3k_4\,[12\,(2+3e^2)(3-30\theta'^2+35\theta^4)-120e^2\,(1-8\theta'^2+7\theta'^4)\ \mathrm{Cos}\,(2u-2f)]\Big)$

$+[(2+e\ \mathrm{Cos}\, f)\ \mathrm{Cos}\, f+e\,]\times(A)-\dfrac{\mathrm{Sin}\, f}{e}\,(2+e\ \mathrm{Cos}\, f)+(B)+(C)$ ... (18)

Where (C) is given by

$(C)=3k_2^2\{96\gamma^2\ [-5(7-18\theta'^2-77\theta'^4)+\eta'^2\ (35-162\theta'^2+55\theta^4)]+192e^2\gamma\,(7-144\theta'^2+165\theta'^4)\ \mathrm{Cos}\,(2u-2f)+48/e\,[9\eta'^{-2}\,(77-270\theta'^2+297\theta'^4)-8(119-342\theta'^2+121\theta'^4)-4\eta'\ (3+\eta'^2)\times(7-54\theta'^2+99\theta'^4)+\eta'^2\ (371-1170\theta'^2-121\theta'^4)]\ \mathrm{Sin}\, f+6\,[87\eta'^{-2}(77-270\theta'^2+297\theta'^4)-2(2723-7506\theta'^2+4807\theta'^4)$
$-64\eta'\,(7-54\theta'^2+99\theta'^4)+\eta'^2\,(651-2178\theta'^2+847\theta'^4)]\mathrm{Sin}\,2f+8e[41\eta'^{-2}\times(77-270\theta'^2+297\theta'^4)-(1337-3510\theta'^2+2277\theta'^4)-8\eta'(7-54\theta'^2+99\theta'^4)]\times$
$\mathrm{Sin}\,3f+24e^2\,[5\eta'^{-2}\,(77-270\theta'^2+297\theta'^4)G8-(7-18\theta'^2+11\theta'^4)]\ \mathrm{Sin}\,4f+24e^3\eta'^{-2}\,(77-270\theta'^2+297\theta'^4)\ \mathrm{Sin}\,5f+2e^4\eta'^{-2}(77-270\theta'^2+297\theta'^4)\ \mathrm{Sin}\,6f$
$-12e^4\eta'^{-2}\,(7-36\theta'^2+33\theta'^4)\times\mathrm{Sin}(6f-2u)-144e^3\eta'^{-2}(7-36\theta'^2+33\theta'^4)\times\mathrm{Sin}\,(5f-2u)+48e^2\,(7-36\theta'^2+33\theta'^4)(-15\eta'^{-2}+2)\ \mathrm{Sin}\,(4f-2u)+48e\,(7-36\theta'^2+33\theta'^4)\,(-41\eta'^{-2}+15+2\eta')\ \mathrm{Sin}(3f-2u)+96/e\,[(7-36\theta'^2+33\theta'^4)\times(27\eta'^{-2}-9\eta'+5\eta'^3)-(175-252\theta'^2+33\theta'^4)+2\eta'^2\,(7+288\theta'^2-363\theta'^4)]$

$\mathrm{Sin}(2u-f)+384[4\eta'^2\theta'^2(9-11\theta'^2)-(7-54\theta'^2-77\theta'^4)]\mathrm{Sin}2u+\dfrac{32}{e}[(7-36\theta'^2+$

$+33\theta'^4)(-31\eta'^{-2}+27\eta'+\eta'^2)+3\,(77-463\theta'^2+451\theta'^4)-2\eta'^2(7-216\theta'^2+$
$253\theta'^4)]\times \mathrm{Sin}\,(2u+f)+12\,[3\,(7-35\theta'^2+33\theta'^4)(-37\eta'^{-2}+15\eta')+2\,(357-$
$1901\theta'^2+1787\theta'^4)-\eta'^2\,(7-183\theta'^2+209\theta'^4)]\ \mathrm{Sin}(2u+2f)+48e\,(7-35\theta'^2+$
$33\theta'^4)\,(-41\eta'^{-2}+7+2\eta')\ \mathrm{Sin}\,(2u+3f)+16e^2\,(7-36\theta'^2+33\theta'^4)\,(-45\eta'^2+$
$2)\ \mathrm{Sin}\ (2u+4f)-144e^3\eta'^{-2}\times(7-35\theta'^2+33\theta'^4)\ \ \mathrm{Sin}\ (2u+5f)-12e^4\eta'^{-2}$
$(7-36\theta'^2+33\theta'^4)\times \mathrm{Sin}\,(2u+6f)+9e^4\eta'^{-2}\,(7-18\theta'^2+11\theta'^4)\ \mathrm{Sin}(6f-4u)+$
$108e^3\eta'^{-2}\,(7-18\theta'^2+11\theta'^4).\mathrm{Sin}\,(5f-4u)+36e\,(-41\eta'^{-2}+29)\times(7-18\theta'^2+$
$11\theta'^4)\ \mathrm{Sin}\,(4u-3f)+3\,[-783\eta'^{-2}\ \ (7-18\theta'^2+11\theta'^4)+2\,(2345-5156\theta'^2+$
$2629\theta'^4)+\eta'^2\,(455-2898\theta'^2+2827\theta'^4)]\times \mathrm{Sin}\,(4u-2f)+\frac{24}{e}\,[-81\eta'^{-2}\ \ (7-$
$18\theta'^2+11\theta'^4)+16\,(28-45\theta'^2+11\theta'^4)+\eta'^2\,(119-738\theta'^2+715\theta'^4)]\ \mathrm{Sin})\ (4u$
$-f)+24\,[-(357-1278\theta'^2+1001\theta'^4)+\eta'^2\,(77-342\theta'^2+297\theta'^4)\ \ \mathrm{Sin}\ \ 4u+$
$\frac{24}{e}\,[81\eta'^{-2}\,(7-18\theta'^2+11\theta'^4)-8\,(105-283\theta'^2+187\theta'^4)+\eta'^2\ \ (273-845\theta'^2$
$+605\theta'^4)]\ \mathrm{Sin}\,(4u+f)+[2349\eta'^{-2}\,(7-18\theta'^2+11\theta'^4)-2\,(6027-15786\theta'^2+$
$9823\theta'^4)+\eta'^2(1099-3402\theta'^2+2431\theta'^4)]\mathrm{Sin}(4u+2f)+36e(7-18\theta'^2+11\theta'^4)\times$
$(41\eta'^{-2}-13)\times \mathrm{Sin}\,(4u+3f)+12e^2\,(7-18\theta'^2+11\theta'^4)\ \ (45\eta'^{-2}-4)\ \mathrm{Sin}\,(4u+$
$4f)+108e^3\eta'^{-2}\,(7-18\theta'^2+11\theta'^4)\ \ \ \mathrm{Sin}\ \ (4u+5f)+9e^4\eta'^2\,(7-18\theta'^2+11\theta'^4)$
$\times \mathrm{Sin}(4u+6f)]+32k_4\,\{2(21-270\theta'^2+335\theta'^4)[6\gamma\,(2+3e^2)+9(4e+e^3)\,\mathrm{Sin}f+$
$9e^2\ \mathrm{Sin}\,2f+e^3\ \ \mathrm{Sin}\,3\,f\,]-2\,(7-720\theta'^2+770\theta'^4)\times[60\gamma e^2\ \mathrm{Cos}\ \ (2u-2f)+30$
$(4e+e^3)\ \mathrm{Sin}\,(2u-f)+20\,(2+3e^2)\ \mathrm{Sin}\,2u+10\ \ \ (4e+e^3)\ \mathrm{Sin}\ \ (2u+f)+15e^2$
$\mathrm{Sin}\ \ (2u+2f\ \ )+2e^3\ \ \ \mathrm{Sin}\ \ \ \ \ (2u+3f\ \ )+10e^3\ \ \ \ \mathrm{Sin}\ \ \ \ (3f-2u)\ \ +\ \ \ (7-$
$18\theta'^2+11\theta'^4)\,[35e^3\ \mathrm{Sin}\,(4u-3f)+105e^2\ \mathrm{Sin}\,(4u-2f)+35\,(4e+e^3)\ \mathrm{Sin}\,(4u$
$-f)+35\,(2+3e^2)\ \mathrm{Sin}\,4u+21\,(4e+e^3)\ \mathrm{Sin}\ \ (4u+f)+35e^2\ \ \mathrm{Sin}\ \ (4u+2f)+$
$5e^3\ \mathrm{Sin}\ \ (4u+3f)]\}$

6. Results Obtained.

In article (4) we have calculated Delaunay variables with the second order perturbation. They are given by (9), (10), (11), (12), (14) and (15). In article (5), we have calculated polar co-ordinates and they are given by (16), (17) and (18).

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